Faculty Of Engineering & Architecture

Academic Year: 2019/2020 Semester: 4

Faculty: Engineering &Architecture (Civil Eng.)

Batch No. :3

Course Title: Numerical methods Course code: MAT221

Lecture Title: Itrative method of solving linear system of equation(Jacobi methd)

Lecture No.:5

Course coordinator: Dr. Ibrahim M.Ibrahim

Contact: 00249917066160 – ataalmanan@yahoo.com

Itrative method of solving linear system of equation

These methods solves the problem $Ax = b$

Undirectly they generate asequence of vectors $\underline{x}_k$, starting with any initial guess $\underline{x}_0$, hopefully converging to the solve $\underline{x}$. $(\underline{x}_0, \underline{x}_1, \underline{x}_2, \cdots)$

General description

$\underline{x}$ Solves, $A\underline{x} = \underline{b}$ if and only if it solves

$$\underline{0} = \underline{b} - A\underline{x}$$

If and only if it solves:

$$C\,\underline{x} = C\,\underline{x} + \underline{b} - A\,\underline{x}$$

Where C is a non – singular matrix

If and only if it solves

$$\underline{x} = C^{-1}\,\underline{b} + C^{-1}\,(C - A)\,\underline{x}$$

To concludes:

$\underline{x}$ is a solution to

$$A\,\underline{x} = \underline{b} \qquad (1)$$

If and only if it is a solution to

$$\underline{x} = \underline{d} + T\,\underline{x} \qquad (2)$$

Where $\underline{d} = C^{-1}\,\underline{b}$, $T = C^{-1}(C - A)$

Now, starting with any initial guess $\underline{x}_0$ and using equation (2) the following sequence is constructed

$$\underline{x}_{k+1} = \underline{d} + T\,\underline{x}_k \qquad (3) \qquad k = 0, 1, 2, \ldots\ldots\ldots$$

Jacobi method

A-Using matrix

It is an iterative method, defined by equation (3) for which

$$C = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

For C to be non-singular a_{ii} must not be zero for any i

$$C^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{a_{nn}} \end{bmatrix}$$

$$\underline{d} = C^{-1}\underline{b} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{a_{nn}} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \frac{b_1}{a_{11}} \\ \vdots \\ \frac{b_i}{a_{ii}} \\ \vdots \\ \frac{b_n}{a_{nn}} \end{bmatrix}$$

$$T = C^{-1}(C - A) = C^{-1}\begin{bmatrix} 0 & -a_{12} & \cdots & & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & & & -a_{2n} \\ \vdots & & 0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ -a_{n1} & \cdots & \cdots & \cdots & -a_{n,n-1} & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & \frac{-a_{12}}{a_{11}} & \cdots & & \cdots & \frac{-a_{1n}}{a_{11}} \\ \frac{-a_{21}}{a_{22}} & 0 & \frac{-a_{23}}{a_{22}} & & & \frac{-a_{2n}}{a_{22}} \\ \vdots & & 0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ \frac{-a_{n1}}{a_{nn}} & \cdots & \cdots & \cdots & \frac{-a_{n,n-1}}{a_{nn}} & 0 \end{bmatrix}$$

That is

$$t_{ij} = \begin{cases} 0 & i = j \\ -\dfrac{a_{ij}}{a_{ii}} & i \neq j \end{cases}$$

Now if $(\underline{x}_{k+1})_i$ is the ith component of $\underline{x}_{k+1}$

Then

$$(\underline{x}_{k+1})_i = \underline{d}_i + (T\,\underline{x}_k)_i$$

Where $(T\,\underline{x}_k)_i$ is the ith component of $T\,\underline{x}_k$

$$\underline{d}_i = \frac{b_i}{a_{ii}}$$

$$(T\,\underline{x}_k)_i = [t_{i1} \quad t_{i2} \quad \cdots \quad t_{in}]$$

$$= \sum_{j=1}^{n} t_{ij}\,(\underline{x}_k)_j$$

$$= \sum_{j=1}^{n} -\frac{a_{ij}}{a_{ii}}(\underline{x}_k)_j$$

$$= -\frac{1}{a_{ii}}\left[\sum_{j=1}^{i-1} a_{ij}(\underline{x}_k)_j + \sum_{j=i+1}^{n} a_{ij}(\underline{x}_k)_j\right]$$

$$(\underline{x}_{k+1})_i = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}(\underline{x}_k)_j - \sum_{j=i+1}^{n} a_{ij}(\underline{x}_k)_j\right]$$

Example

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(\underline{x}_1)_i = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}(\underline{x}_0)_j - \sum_{j=i+1}^{3} a_{ij}(\underline{x}_0)_j\right]$$

I=1

$$(\underline{x}_1)_1 = \frac{1}{a_{11}}\left[b_1 - \sum_{j=1}^{i-1} a_{12}(\underline{x}_0)_2 - \sum_{j=i+1}^{3} a_{ij}(\underline{x}_0)_3\right]$$

$$(\underline{x}_1)_1 = \frac{1}{4}[1 - 1(\underline{x}_0)_2 - 1(\underline{x}_0)_3] = \frac{1}{4}[1 - 0 - 0] = \frac{1}{4}$$

$$(\underline{x}_1)_2 = \frac{1}{4}[1 - 1(\underline{x}_0)_1 - 1(\underline{x}_0)_3] = \frac{1}{4}[1 - 1 - 0] = 0$$

$$(\underline{x}_1)_3 = \frac{1}{4}[1 - 1(\underline{x}_0)_1 - 1(\underline{x}_0)_2] = \frac{1}{4}[1 - 1 - 0] = 0$$

$$\underline{x}_1 = \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \end{bmatrix}$$

B-Using equation

Example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Rewrite the equations, solving for the unknown

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3]$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3]$$

$$x_3 = \frac{1}{a_{33}}[b_3 - a_{31}x_1 - a_{32}x_2]$$

Then, iterate as in fixed point iteration, starting with an initial guess

$$x_1^{(k+1)} = \frac{1}{a_{11}}[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}]$$

Stop until the condition verify

$$\left|x_i^{(k+1)} - x_i^{(k)}\right| \le \varepsilon$$

Where: ε = given tolerance

Example:

Apply the Jacobi method to the system

$3u+v=5$, $u+2v=5$

Solution:

Begin by solving the first equation for u and the second equation for v. we will use the initial guess $(u_0, v_0) = (0, 0)$. We have

$$u = \frac{5-v}{3} \quad , \quad v = \frac{5-u}{2}$$

The two equations are iterated

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/2}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-5/6}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}$$

Further steps of Jacobi show convergence toward the solution, which is [1,2]

The algorithm of Jacobi method

- Read(n,A,b)
- Read(x,e)

- Repeat
- Fori=1tondo
- Yi=bi
- For j=1to i-1 do
- Yi=yi-aij*xj
- Yi=yi/aii
- Diff=y-x
- Fori=1to n do
- Xi=yi
- Until (diff<e)