Faculty Of Engineering & Architecture

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Lecture Title: Introduction to systems of linear equations

Lecture No.:2

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Mathematical back ground

Introduction to systems of linear equations:

- General form of linear equation in the n variables $x_1, x_2, \cdots\cdots, x_n$

$$a_1 x_1 + a_2 x_2 + \cdots\cdots + a_n x_n = b$$

Where $a_1, a_2, \cdots\cdots, a_n$ and b are real constants

$x_1 . x_2, \cdots\cdots, x_n$ = unknowns

- A finite set of linear equations in the variables $x_1 . x_2, \cdots\cdots, x_n$ is called a system of linear equations or a linear system
- A sequence of numbers

$s_1, s_2, \cdots\cdots, s_n$ is called a solution of the system

- Every system of linear equations has no solution, or has exactly one solution, or has infinitely many solutions
- An arbitrary system of m linear equation in n unknowns can be written as

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m$$

Where $x_1, x_2, \cdots, x_n$ are the unknowns

Subscripted a,b denote constants

- The augmented matrix for the system can be writing as

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- The matrix form of this systems are

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Cramer rule for solving linear system

If $Ax = b$ is a system of n linear equations in n unknown such that $\det(A) \neq 0$, then the system has a unique solution . This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\Delta_1}{\Delta} \;,\quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\Delta_2}{\Delta} \;,\quad x_n = \frac{\det(A_n)}{\det(A)} = \frac{\Delta_n}{\Delta}$$

Where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

For example

Let

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots \quad \cdots\cdots\cdots \quad \cdots\cdots\cdots \quad \cdots\cdots\cdots \quad \cdot\cdot$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\det(A_1) = \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\det(A_2) = \Delta_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}$$

$$\det(A_n) = \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}$$

The solution of the system

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\Delta_1}{\Delta} \quad , \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\Delta_2}{\Delta} \quad , \quad x_n = \frac{\det(A_n)}{\det(A)} = \frac{\Delta_n}{\Delta}$$

Example:

Find the solution of the following

system by using the Cramer's rule

$$(i)\quad x_1 + 2x_2 - x_3 = 4$$
$$2x_1 - 3x_2 + x_3 = -1$$
$$5x_1 + 7x_2 + 2x_3 = -1$$

Solution:

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

$$|A| = \Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 5 & 7 & 2 \end{vmatrix} = -40 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 4 & 2 & -1 \\ -1 & -3 & 1 \\ -1 & 7 & 2 \end{vmatrix} = -40$$

$$\Delta_2 = \begin{vmatrix} 1 & 4 & -1 \\ 2 & -1 & -1 \\ 5 & -1 & 2 \end{vmatrix} = 0$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & -3 & -1 \\ 5 & 7 & -1 \end{vmatrix} = 120$$

$$\because x_i = \frac{\Delta_i}{\Delta}$$

$$\therefore x_1 = \frac{\Delta_1}{\Delta} = \frac{-40}{-40} = 1 \;,\; x_2 = \frac{\Delta_2}{\Delta} = \frac{0}{-40} = 0$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{120}{-40} = -3 \quad , \quad u = (1, 0, -3)$$