Faculty Of Engineering & Architecture

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Lecture Title: Matrices and matrix operations

Lecture No.:1

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Mathematical back ground

Matrices and matrix operations:

- A matrix is arectangular array of numbers. The numbers in the array are called the entries in the matrix
- The size of a matrix is described interms of the number of rows and columns

A general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ & & & & & \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$$

- Two matrices are defined to be

equal if they have the same size and their corresponding entries are equal.

In matrix notation

$$A_{m\times n} = [a_{ij}] \quad , \quad B_{m\times n} = [b_{ij}]$$

$$A = B \Leftrightarrow a_{ij} = b_{ij}$$

Example

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix} \quad , \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

If x=5 then A=B

- If A and B are matrices of the same size , then the sum (A+B) is the matrix obtained by adding the entries of B to the corresponding entries of A, and the difference (A-B) is the matrix obtained by subtracting the entries of B from the corresponding entries of A.

- Matrices of different size cannot be added or subtracted.

Example

Let $A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & 1 \\ 3 & 2 & -4 & 5 \end{bmatrix}$

- If A is any matrix and c is any scalar, then the product c A is the matrix obtained by multiplying each entry of the matrix A by c

In notation

$$(cA_{ij}) = c(A_{ij}) = ca_{ij}$$

Example

Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}$, $2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$

- If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows. To find the entry in row I and column j of AB, single out row I from the matrix A and column j from the matrix B.

Multiply the corresponding entries from the row and column together , and then add up the resulting products

Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$(AB)_{2\times 4} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

- If A is any $m \times n$ matrix, then the transpose of A. denoted by A^T , is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A; that is the first column of A^T is the first row of A. and the second column of A^T is the second row of A, and so forth.

Example

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \quad , \quad A^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

- **Elementary row operation**

  + Multiply row I by c $\neq 0$

  + Interchange rows i and j

  + Add c times row i to row j

- **The matrix**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Is invertible if ad-bc $\neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad , \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Using row operation to find A^{-1}

To find the inverse of an invertible matrix A. we must find a sequence of elementary row operation that reduces A To the identity and then perform this same sequence of operation on I_n to obtain A^{-1}

$$[A|I]$$

$$[I|A^{-1}]$$

Example: find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution:

We want to reduce A to the identity matrix by row operations and simultaneously apply these operation to I to produce A^{-1} . The computations are as follows

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -R_1 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$2R_2 + R_3 \rightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \quad -R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad \begin{array}{l} 3R_3 + R_2 \rightarrow R_2 \\ -3R_3 + R_1 \rightarrow R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$-2R_2 + R_1 \rightarrow R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

- ## Diagonal matrices

- A square matrix in which all the entries off the main diagonal are zero is called a diagonal matrix

Example

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- A square matrix in which all the entries above the main diagonal are zero is called lower triangular , and a square matrix in which all the entries below the main diagonal are zero is called upper triangular

Example

$$U=\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \quad , \quad L=\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

- A square matrix $A=[a_{ij}]$ is upper triangular if and only if $a_{ij}=0$ for $i>j$
- A square matrix $A=[a_{ij}]$ is lower triangular if and only if $a_{ij}=0$ for $i<j$
- The transpose of a lower triangular matrix is upper triangular , and the transpose of an upper triangular matrix is lower triangular
- The product of lower

triangular matrices is lower triangular , and the product of upper triangular matrices is upper triangular

- A triangular matrix is invertible if and only if its diagonal entries are all non zero
- The inverse of an invertible lower triangular matrix is lower triangular , and the inverse of an invertible upper triangular matrix is upper triangular
- A square matrix A is called

symmetric if $A = A^T$

Example

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

- A matrix $A = [a_{ij}]$ is a symmetric if and only if $a_{ij} = a_{ji}$ for all values of i and j

Determinants by cofactor:

If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the sub matrix that remains after the ith row and jth column are deleted from A.

The number $(-1)^{i+j} M_{ij}$ is denoted by c_{ij} and is called the cofactor of entry a_{ij}

Example:

Let $$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is $M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$

The cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$

Similarly, the minor of entry a_{32} is $M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$

The cofactor of a_{32} is $C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$

Note:

- The cofactor and minor of an element a_{ij} differ only in sign $\Rightarrow c_{ij} = \pm M_{ij}$
- Aquick way to determine whether to use + or –

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$c_{11} = m_{11}$, $c_{21} = -m_{21}$, $c_{12} = -m_{12}$, $c_{22} = m_{22}$ and so on

- The determinant of nxn matrix can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products , that is for each $1 \le i \le n$ and $1 \le j \le n$

$$\det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots\cdots + a_{in}c_{in}$$

(along the ith row)

$$\det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots\cdots + a_{nj}c_{nj}$$

(along the jth column)

Example

Let $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ evaluate

det(A) by cofactor expansion a long the

first row of A

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

$$c_{11} = -4 \quad , \quad c_{12} = 11$$

$$\det(A) = 3(-4) + 1(11) + 0 = -1$$

$$\det(A) = a_{11}c_{11} + a_{21}c_{21} + a_{31}c_{31}$$

$$c_{11} = -4 \,,\, c_{21} = 2 \,,\, c_{31} = 3$$

$$\det(A) = 3(-4) + (-2)(2) + 5(3) = -1$$

Example

Let
$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Column 2

$$\det(A) = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32} + a_{42}c_{42}$$

$$\det(A) = 1.\begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1.-2.\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1+2) = -6$$

- If A is any nxn matrix and c_{ij} is the cofactor of a_{ij} ,then the matrix

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

Is called the matrix of cofactors from A. the transpose of this matrix is called the adjoint of A and is denoted by adj (A)

Example

Let $$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The cofactors of A are

$$c_{11} = 12 \quad c_{12} = 6 \quad c_{13} = -16$$
$$c_{21} = 4 \quad c_{22} = 2 \quad c_{23} = 16$$
$$c_{31} = 12 \quad c_{32} = -10 \quad c_{33} = 16$$

So the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

And the adjoint of A is

$$adj\,(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

- The square matrix A is invertible if and only if det(A) is not zero

Inverse of a matrix using its adjoint:

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example

Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ find A^{-1}

Det(A)=64

$$A^{-1} = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

- If A is an nxn triangular matrix (upper , lower or diagonal) then det(A) is the product of the entries on the main diagonal of the matrix, that is

$$\det(A) = a_{11} a_{22} \cdots\cdots a_{nn}$$