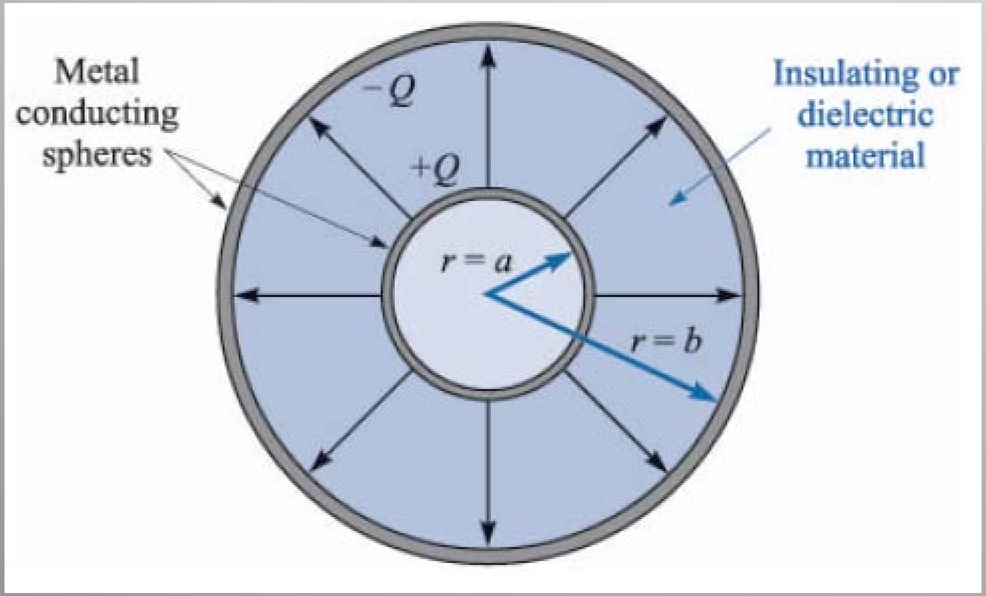
ELECTRIC FLUX DENSITY, GAUSS’ LAW, AND DIVERGENCE

Electric Flux Density

- About 1837, the Director of the Royal Society in London, Michael Faraday, was interested in static electric fields and the effect of various insulating materials on these fields.
- This is the lead to his famous invention, the electric motor.
- He found that if he moved a magnet through a loop of wire, an electric current flowed in the wire. The current also flowed if the loop was moved over a stationary magnet.
- ► Changing magnetic field produces an electric field.

Electric Flux Density

- In his experiments, Faraday had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamed together.
- He also prepared shells of insulating material (or *dielectric* material), which would occupy the entire volume between the concentric spheres.



Electric Flux Density

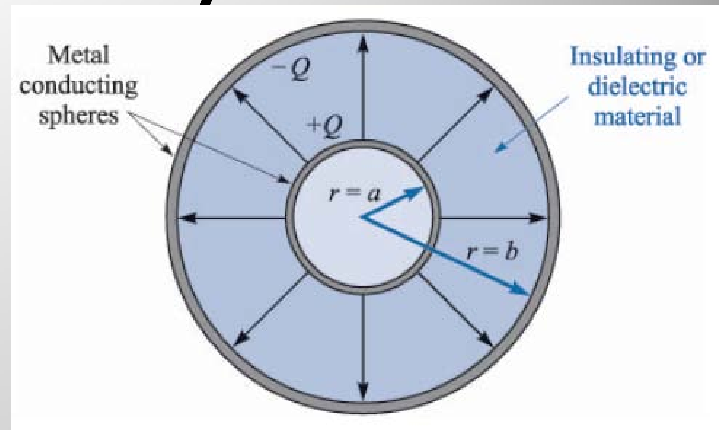
- Faraday found out, that there was a sort of "charge displacement" from the inner sphere to the outer sphere, which was independent of the medium.
- We refer to this flow as displacement, displacement flux, or simply electric flux.

$$\psi = Q$$

- Where ψ is the electric flux, measured in coulombs, and Q is the total charge on the inner sphere, also in coulombs.

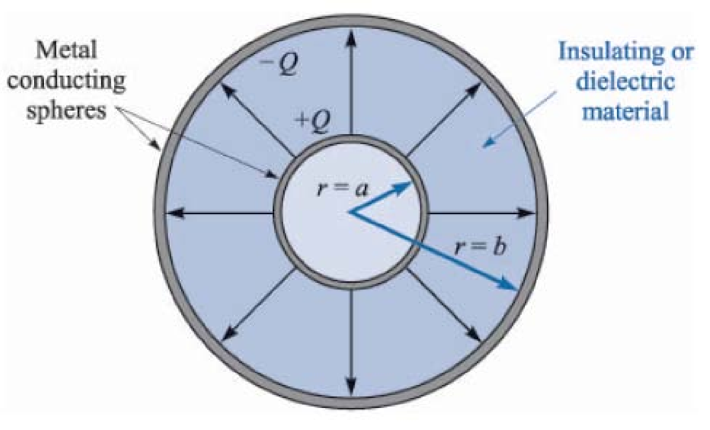
Electric Flux Density

- At the surface of the inner sphere, ψ coulombs of electric flux are produced by the given charge Q coulombs, and distributed uniformly over a surface having an area of $4\pi a^2$ m^2.
- The density of the flux at this surface is $\psi/4\pi a^2$ or $Q/4\pi a^2$ C/m^2.



- The new quantity, *electric flux density*, is measured in C/m^2 and denoted with D.
- The direction of D at a point is the direction of the flux lines at that point.
- The magnitude of D is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Electric Flux Density



- Referring again to the concentric spheres, the electric flux density is in the radial direction :

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

- At a distance *r*, where $a \le r \le b$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

- If we make the inner sphere smaller and smaller, it becomes a point charge while still retaining a charge of *Q*. The electrix flux density at a point *r* meters away is still given by:

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r}$$

Electric Flux Density

- Comparing with the previous chapter, the radial electric field intensity of a point charge in free space is:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r$$

- Therefore, in free space, the following relation applies:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

- For a general volume charge distribution in free space:

$$\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\varepsilon_0 R^2}\mathbf{a}_R$$

$$\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2}\mathbf{a}_R$$

Electric Flux Density

- Example
  Find the electric flux density at a point having a distance 3 m from a uniform line charge of 8 nC/m lying along the *z* axis in free space.

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho \quad \Rightarrow \mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho = \frac{8\times10^{-9}}{2\pi\rho}\mathbf{a}_\rho = \frac{1.273\times10^{-9}}{\rho}\mathbf{a}_\rho \ \mathrm{C/m^2}$$

For the value ρ = 3 m,

$$\mathbf{D} = \frac{1.273\times10^{-9}}{3} = 4.244\times10^{-10}\mathbf{a}_\rho \ \mathrm{C/m^2} = \underline{\underline{0.424\mathbf{a}_\rho \ \mathrm{nC/m^2}}}$$

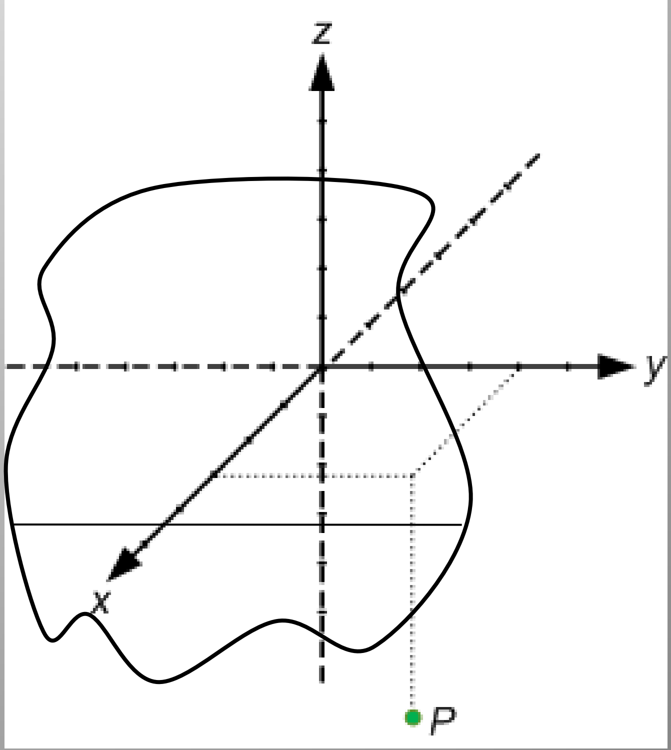
Electric Flux Density

- <u>Example</u>
  Calculate D at point *P*(6,8,–10) produced by a uniform surface charge density with ρ_s = 57.2 μC/m^2 on the plane x = 9.

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon_0}\mathbf{a}_N \quad \Rightarrow \mathbf{D} = \frac{\rho_s}{2}\mathbf{a}_N = \frac{57.2\times10^{-6}}{2}\mathbf{a}_N = 28.6\mathbf{a}_N \ \mu\mathrm{C/m^2}$$

At *P*(6,8,–10),

$$\mathbf{a}_N = -\mathbf{a}_x \qquad \Rightarrow \mathbf{D} = \underline{\underline{-28.6\mathbf{a}_x \ \mu\mathrm{C/m^2}}}$$



Gauss's Law

- The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface.

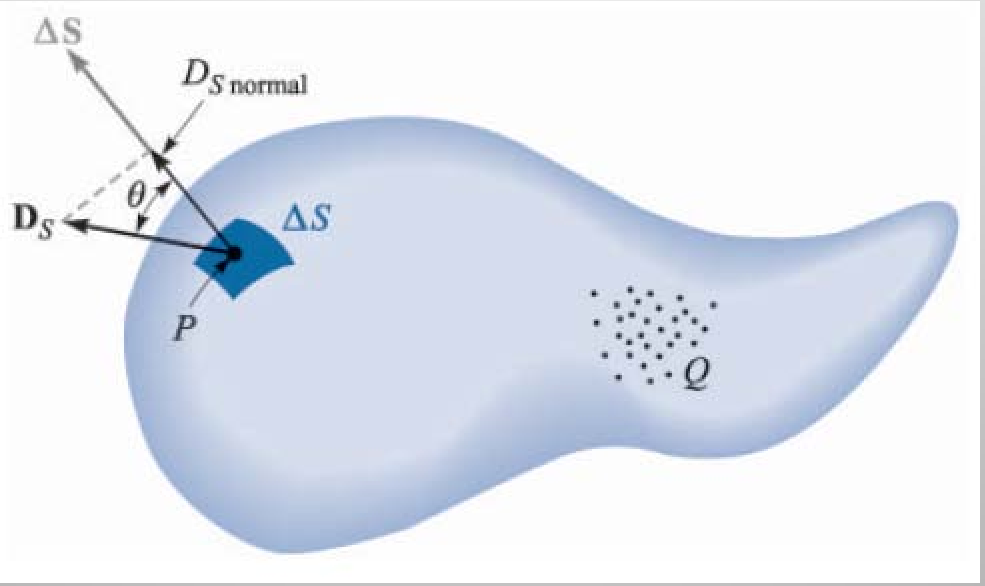
$$\psi = Q$$

- Faraday's experiment can be generalized to the following statement, which is known as Gauss's Law:

  "The electric flux passing through any closed surface is equal to the total charge enclosed by that surface."

Gauss's Law

- Imagine a distribution of charge, shown as a cloud of point charges, surrounded by a closed surface of any shape.



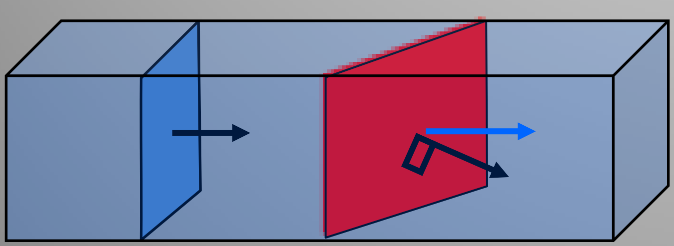
- If the total charge is *Q*, the *Q* coulombs of electric flux will pass through the enclosing surface.
- At every point on the surface the electric-flux-density vector D will have some value D_S (subscript *S* means that D must be evaluated at the surface).

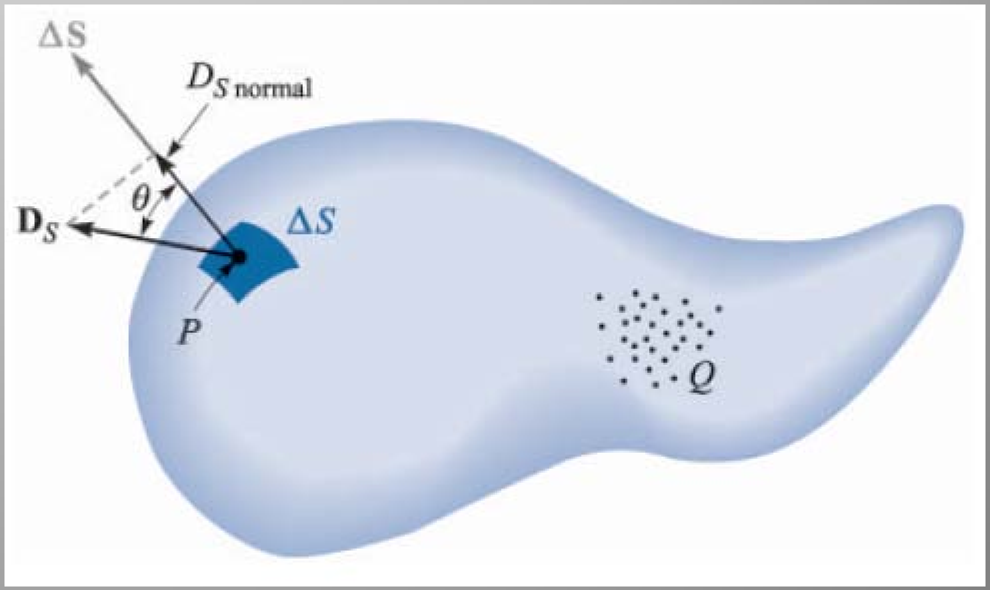
Gauss's Law

- ΔS defines an incremental element of area with magnitude of Δ*S* and the direction normal to the plane, or tangent to the surface at the point in question.
- At any point *P*, where D_S makes an angle *θ* with ΔS, then the flux crossing ΔS is the product of the normal components of D_S and ΔS.

$$\Delta\psi = \text{flux crossing } \Delta S = D_S \cos\theta \cdot \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

$$\psi = \int d\psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$





Gauss's Law

- The resultant integral is a closed surface integral, with *d*S always involves the differentials of two coordinates
  ► The integral is a double integral.
- We can formulate the Gauss's law mathematically as:

$$\psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

- The charge enclosed meant by the formula above might be several point charges, a line charge, a surface charge, or a volume charge distribution.

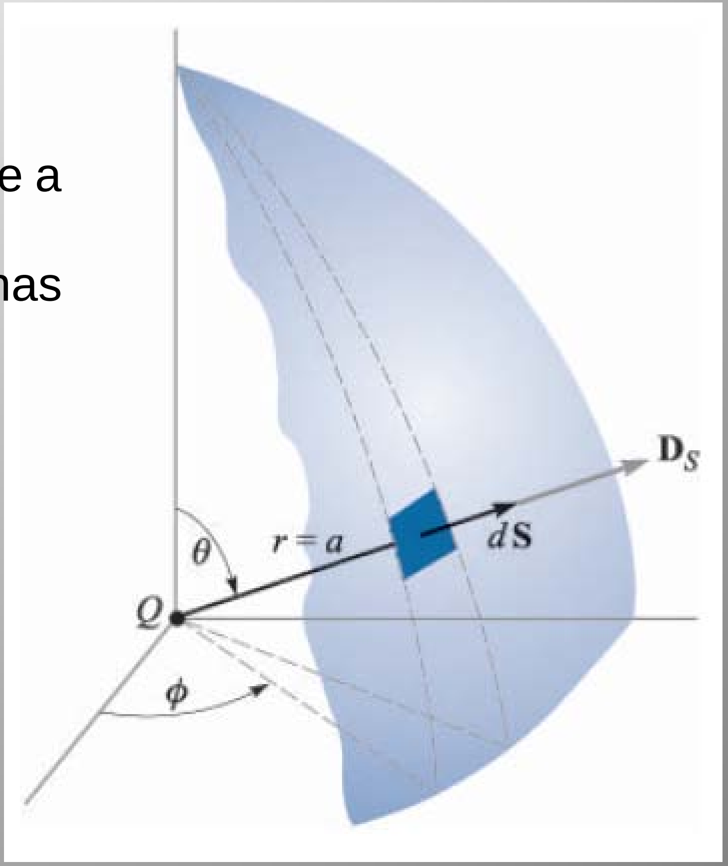
$$Q = \sum Q_n \qquad Q = \int \rho_L dL \qquad Q = \int_S \rho_S dS \qquad Q = \int_{vol} \rho_v dv$$

Gauss's Law

- We now take the last form, written in terms of the charge distribution, to represent the other forms:

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv$$

- Illustration. Let a point charge *Q* be placed at the origin of a spherical coordinate system, and choose a closed surface as a sphere of radius *a*.
- The electric field intensity due to the point charge has been found to be:



$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \;\Rightarrow\; \mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

Gauss's Law

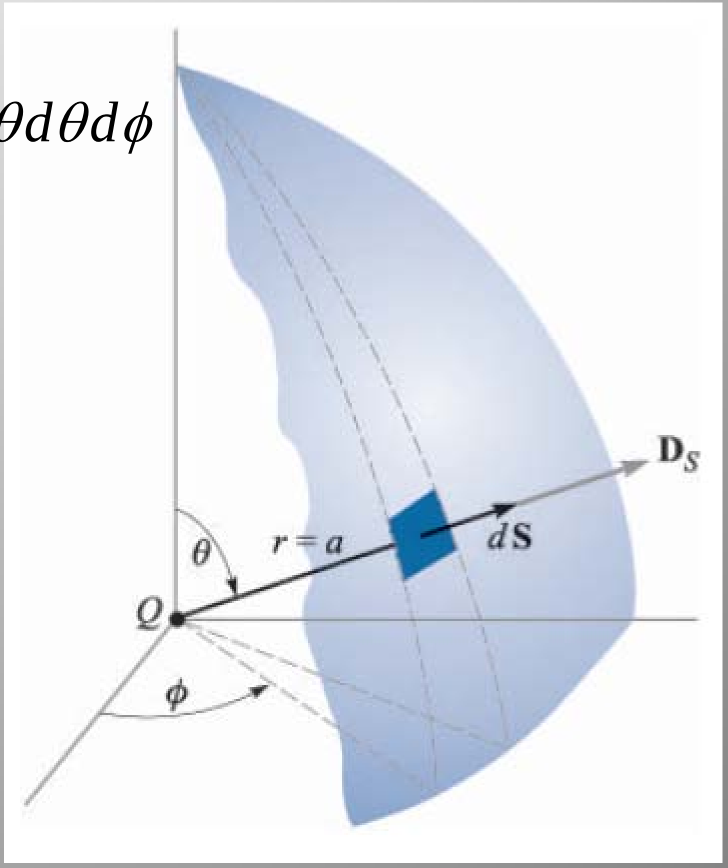
- At the surface, $r = a$,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

$$d\mathbf{S} = a^2 \sin\theta \, d\theta d\phi \, \mathbf{a}_r$$

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta d\theta d\phi \, \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi}\sin\theta d\theta d\phi$$

$$\begin{aligned}\Psi &= \oint_S \mathbf{D}_S \cdot d\mathbf{S} \\ &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{Q}{4\pi}\sin\theta d\theta d\phi\bigg|_{r=a} \\ &= -\frac{Q}{4\pi}\cos\theta\bigg|_{\theta=0}^{\pi}\theta\bigg|_{\phi=0}^{2\pi} \\ &= \underline{\underline{Q}}\end{aligned}$$



Application of Gauss's Law: Some Symmetrical Charge Distributions

- Let us now consider how to use the Gauss's law to calculate the electric field intensity D_S:

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

- The solution will be easy if we are able to choose a closed surface which satisfies two conditions:
  1. D_S is everywhere either normal or tangential to the closed surface, so that $D_S \cdot dS$ becomes either $D_S dS$ or zero, respectively.
  2. On that portion of the closed surface for which $D_S \cdot dS$ is not zero, D_S is constant.

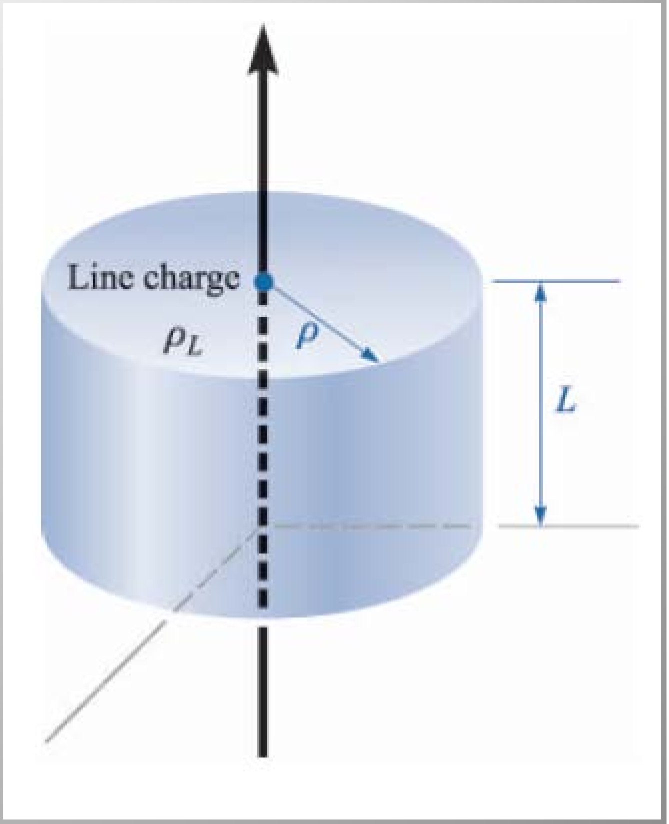
- For point charge ► The surface of a sphere.
- For line charge ► The surface of a cylinder.

Application of Gauss's Law: Some Symmetrical Charge Distributions

- From the previous discussion of the uniform line charge, only the radial component of D is present:

$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

- The choice of a surface that fulfill the requirement is simple: a cylindrical surface.
- D_ρ is every normal to the surface of a cylinder. It may then be closed by two plane surfaces normal to the *z* axis.



Application of Gauss's Law: Some Symmetrical Charge Distributions

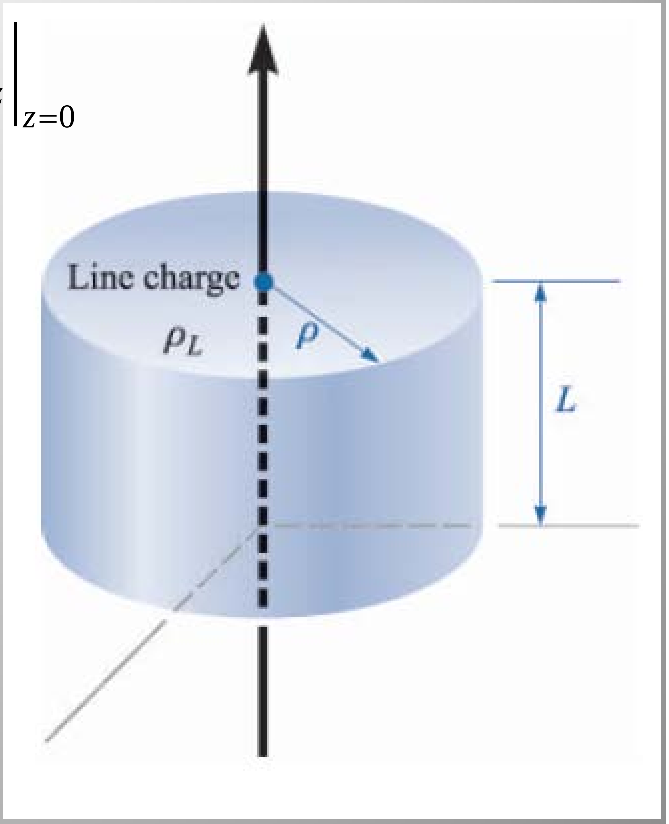
$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

$$= D_\rho \int_{\text{sides}} dS_\rho \Big|_{\rho=\rho'} + D_z \int_{\text{top}} dS_z \Big|_{z=L} + D_z \int_{\text{bottom}} dS_z \Big|_{z=0}$$

$$= D_\rho \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz$$

$$= D_\rho 2\pi\rho L$$

$$\Rightarrow D_\rho = \frac{Q}{2\pi\rho L}$$



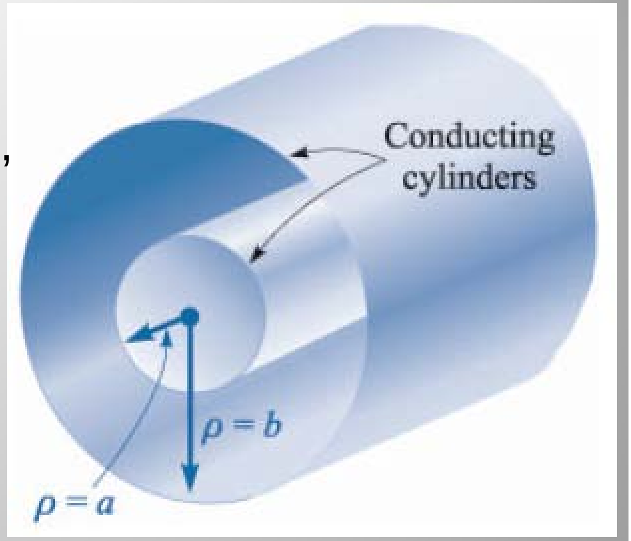
- We know that the charge enclosed is $\rho_L L$,

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

$$E_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}$$

Application of Gauss's Law: Some Symmetrical Charge Distributions

- The problem of a coaxial cable is almost identical with that of the line charge.
- Suppose that we have two coaxial cylindrical conductors, the inner of radius *a* and the outer of radius *b*, both with infinite length.
- We shall assume a charge distribution of ρ_S on the outer surface of the inner conductor.



- Choosing a circular cylinder of length *L* and radius ρ, $a < \rho < b$, as the gaussian surface, we find:

$$Q = D_S 2\pi\rho L$$

- The total charge on a length *L* of the inner conductor is:

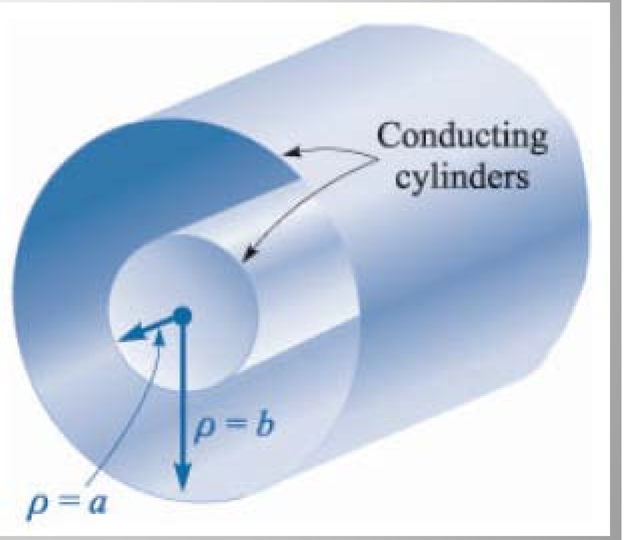
$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_S a\, d\phi\, dz = 2\pi a L \rho_S \Rightarrow \boxed{D_S = \frac{a\rho_S}{\rho}}$$

Application of Gauss's Law: Some Symmetrical Charge Distributions

- For one meter length, the inner conductor has $2\pi a\rho_S$ coulombs, hence $\rho_L = 2\pi a\rho_S$,

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

- Everly line of electrix flux starting from the inner cylinder must terminate on the inner surface of the outer cylinder:



$$Q_{\text{outer cyl}} = -2\pi aL\rho_{S,\text{inner cyl}}$$

$$2\pi bL\rho_{S,\text{outer cyl}} = -2\pi aL\rho_{S,\text{inner cyl}}$$

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b}\rho_{S,\text{inner cyl}}$$

- If we use a cylinder of radius $\rho > b$, then the total charge enclosed will be zero.
  ► There is no external field,

$$D_S = 0$$

- Due to simplicity, noise immunity and broad bandwidth, coaxial cable is still the most common means of data transmission over short distances.

Application of Gauss's Law: Some Symmetrical Charge Distributions

- Example
  A 50-cm length of coaxial cable has an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. Find the charge density on each conductor and the expressions for E and D fields.

$$Q_{\text{inner cyl}} = 2\pi a L \rho_{S,\text{inner cyl}}$$

$$\Rightarrow \rho_{S,\text{inner cyl}} = \frac{Q_{\text{inner cyl}}}{2\pi a L} = \frac{30\times10^{-9}}{2\pi(10^{-3})(0.5)} = \underline{\underline{9.55\,\mu\text{C}/\text{m}^2}}$$

$$Q_{\text{outer cyl}} = 2\pi b L \rho_{S,\text{outer cyl}} = -Q_{\text{inner cyl}}$$

$$\Rightarrow \rho_{S,\text{outer cyl}} = \frac{-Q_{\text{inner cyl}}}{2\pi b L} = \frac{-30\times10^{-9}}{2\pi(4\times10^{-3})(0.5)} = \underline{\underline{-2.39\,\mu\text{C}/\text{m}^2}}$$

Application of Gauss's Law: Some Symmetrical Charge Distributions

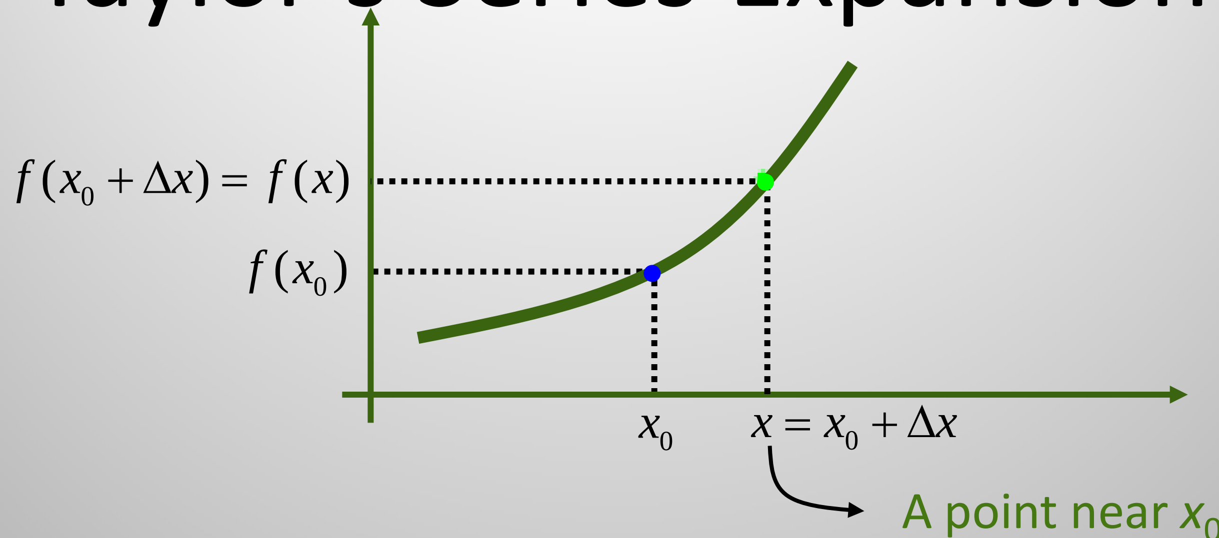
$$D_\rho = a\frac{\rho_{S,\text{inner cyl}}}{\rho} = 10^{-3}\frac{(9.55\times10^{-6})}{\rho} = \underline{\underline{\frac{9.55}{\rho}\,\text{nC/m}^2}}$$

$$E_\rho = \frac{D_\rho}{\varepsilon_0} = \frac{9.55\times10^{-9}}{8.854\times10^{-12}\rho} = \underline{\underline{\frac{1079}{\rho}\,\text{V/m}}}$$

Application of Gauss's Law: Differential Volume Element

- We are now going to apply the methods of Gauss's law to a slightly different type of problem: a surface without symmetry.
- We have to choose such a very small closed surface that D is almost constant over the surface, and the small change in D may be adequately represented by using the first two terms of the Taylor's-series expansion for D.
- The result will become more nearly correct as the volume enclosed by the gaussian surface decreases.

Taylor's Series Expansion



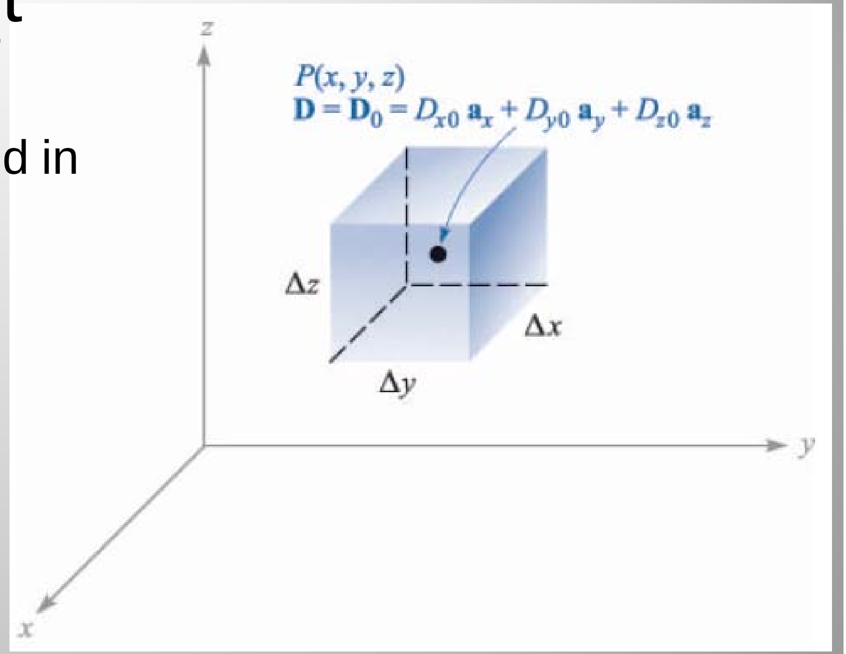
$$f(x) = f(x_0 + \Delta x)$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \cdots + \frac{f^n(x_0)}{n!}(\Delta x)^n$$

Only the linear terms are used for the linearization

Application of Gauss's Law: Differential Volume Element

- Consider any point *P*, located by a rectangular coordinate system.
- The value of D at the point *P* may be expressed in rectangular components:



$$\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$$

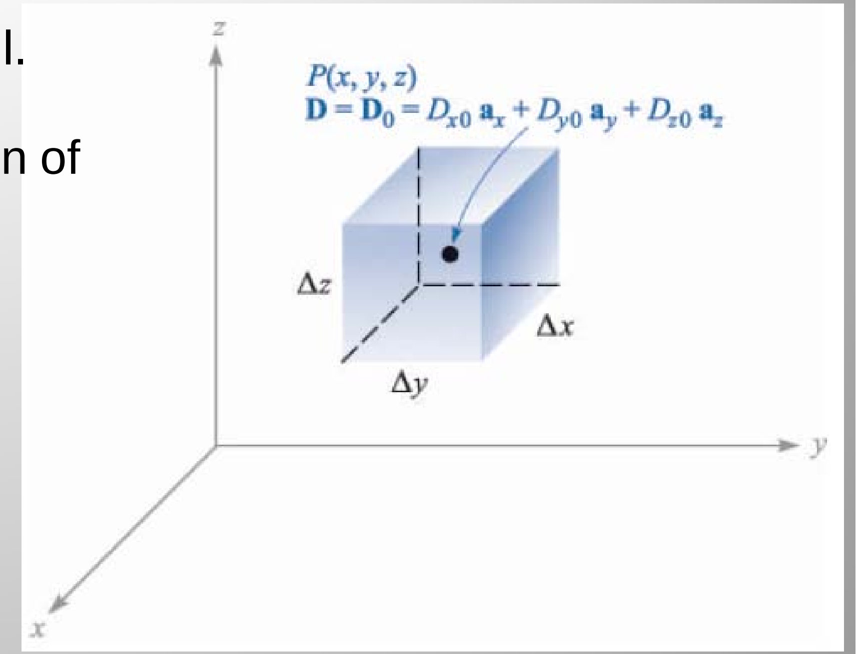
- We now choose as our closed surface, the small rectangular box, centered at *P*, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Application of Gauss's Law: Differential Volume Element

- We will now consider the front surface in detail.
- The surface element is very small, thus D is essentially constant over this surface (a portion of the entire closed surface):



$$\int_{\text{front}} \text{B}\, \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}}$$

$$\text{B}\, \mathbf{D}_{\text{front}} \cdot \Delta y \Delta z\ \mathbf{a}_x$$

$$\text{B}\, D_{x,\text{front}} \Delta y \Delta z$$

- The front face is at a distance of Δ*x*/2 from *P*, and therefore:

$$D_{x,\text{front}}\ \text{B}\, D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$\text{B}\, D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

Application of Gauss's Law: Differential Volume Element

- We have now, for front surface:

$$\int_{\text{front}} \mathrm{B} \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

- In the same way, the integral over the back surface can be found as:

$$\int_{\text{back}} \mathrm{B}\, \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}}$$

$$\mathrm{B}\, \mathbf{D}_{\text{back}} \cdot (-\Delta y \Delta z\ \mathbf{a}_x)$$

$$\mathrm{B} -D_{x,\text{back}} \Delta y \Delta z$$

$$D_{x,\text{back}}\ \mathrm{B}\ D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{\text{back}} \mathrm{B} \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

Application of Gauss's Law: Differential Volume Element

- If we combine the two integrals over the front and back surface, we have:

$$\int_{\text{front}} + \int_{\text{back}} \mathrm{B} \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z$$

- Repeating the same process to the remaining surfaces, we find:

$$\int_{\text{right}} + \int_{\text{left}} \mathrm{B} \frac{\partial D_y}{\partial y} \Delta y \Delta x \Delta z$$

$$\int_{\text{top}} + \int_{\text{bottom}} \mathrm{B} \frac{\partial D_z}{\partial z} \Delta z \Delta x \Delta y$$

- These results may be collected to yield:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \ \mathrm{B} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \ \mathrm{B} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

Application of Gauss's Law: Differential Volume Element

- The previous equation is an approximation, which becomes better as Δv becomes smaller.
- For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv, with the result:

$$\text{Charge enclosed in volume } \Delta v \;\text{B}\; \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \Delta v$$

Application of Gauss's Law: Differential Volume Element

- <u>Example</u>

  Let D = y^2z^3 $\mathbf{a}_x$ + $2xyz^3$ $\mathbf{a}_y$ + $3xy^2z^2$ $\mathbf{a}_z$ pC/m^2 in free space.
  (*a*) Find the total electric flux passing through the surface x = 3, 0 ≤ y ≤ 2, 0 ≤ z ≤ 1 in a direction away from the origin. (*b*) Find |E| at *P*(3,2,1). (*c*) Find the total charge contained in an incremental sphere having a radius of 2 μm centered at *P*(3,2,1).

(*a*)

$$\psi = \int_S \mathbf{D}_S \cdot d\mathbf{S}$$

$$= \int_{z=0}^{1}\int_{y=0}^{2}\left(y^2z^3\mathbf{a}_x + 2xyz^3\mathbf{a}_y + 3xy^2z^2\mathbf{a}_z\right)\cdot\left(dydz\ \mathbf{a}_x\right)\Big|_{x=3}$$

$$= \int_0^1\int_0^2 y^2z^3dydz$$

$$= \tfrac{1}{3}y^3\Big|_0^2 \tfrac{1}{4}z^4\Big|_0^1$$

$$= \tfrac{2}{3}\,\text{pC}$$

Application of Gauss's Law: Differential Volume Element

(*b*) $\mathbf{D} = y^2z^3\mathbf{a}_x + 2xyz^3\mathbf{a}_y + 3xy^2z^2\mathbf{a}_z$

$$\mathbf{D}_P = (2)^2(1)^3\mathbf{a}_x + 2(3)(2)(1)^3\mathbf{a}_y + 3(3)(2)^2(1)^2\mathbf{a}_z$$

$$= 4\mathbf{a}_x + 12\mathbf{a}_y + 36\mathbf{a}_z\ p\text{C/m}^2$$

$$\left|\mathbf{D}_P\right| = D_P = \sqrt{(4)^2 + (12)^2 + (36)^2}$$

$$= 38.158\ p\text{C/m}^2$$

$$\left|\mathbf{E}_P\right| = \frac{\left|\mathbf{D}_P\right|}{\varepsilon_0}$$

$$= \frac{38.158\ p\text{C/m}^2}{8.854\times10^{-12}}$$

$$= \underline{\underline{4.31\,\text{V/m}}}$$

Application of Gauss's Law: Differential Volume Element

(c)

$$Q \text{ B} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

$$Q_P \text{ B} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Bigg|_P \Delta v$$

$$\text{B} \left(0 + 2xz^3 + 6xy^2 z \right) \Big|_{\substack{x=3 \\ y=2 \\ z=1}} \text{pC/m}^3 \times \tfrac{4}{3} \pi (2 \times 10^{-6})^3 \text{m}^3$$

$$\text{B} \left(0 + 2(3)(1)^3 + 6(3)(2)^2 (1) \right) \tfrac{4}{3} \pi (2 \times 10^{-6})^3 \, p\text{C}$$

$$\text{B} \; \underline{\underline{2.61 \times 10^{-27} \text{ C}}}$$

Divergence

- We shall now obtain an exact relationship, by allowing the volume element Δv to shrink to zero.

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right) \approx \frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\frac{Q}{\Delta v}$$

$$\Downarrow$$

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}$$

- The last term is the volume charge density ρ_v, so that:

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\rho_v$$

Divergence

- Let us no consider one information that can be obtained from the last equation:

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v}$$

- This equation is valid not only for electric flux density D, but also to any vector field A to find the surface integral for a small closed surface.

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

Divergence

- This operation received a descriptive name, ***divergence***. The divergence of A is defined as:

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

"The divergence of the vector flux density A is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero."

- A positive divergence of a vector quantity indicates a source of that vector quantity at that point.
- Similarly, a negative divergence indicates a sink.

Divergence

$$\text{div}\,\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Rectangular

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Cylindrical

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Spherical

Divergence

- Example
  If D = $e^{-x}\sin y\ \mathbf{a}_x - e^{-x}\cos y\ \mathbf{a}_y + 2z\ \mathbf{a}_z$, find div D at the origin and $P(1,2,3)$

$$\operatorname{div}\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = -e^{-x}\sin y + e^{-x}\sin y + 2 = \underline{\underline{2}}$$

Regardles of location the divergence of D equals 2 C/m^3.

Maxwell's First Equation (Electrostatics)

- We may now rewrite the expressions developed until now:

$$\operatorname{div} \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v}$$

$$\operatorname{div} \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\operatorname{div} \mathbf{D} = \rho_v$$

Maxwell's First Equation
Point Form of Gauss's Law

- This first of Maxwell's four equations applies to electrostatics and steady magnetic field.
- Physically it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

The Vector Operator ∇ and The Divergence Theorem

- Divergence is an operation on a vector yielding a scalar, just like the dot product.
- We define the del operator ∇ as a vector operator:

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

- Then, treating the del operator as an ordinary vector, we can write:

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\text{div}\ \mathbf{D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

The Vector Operator ∇ and The Divergence Theorem

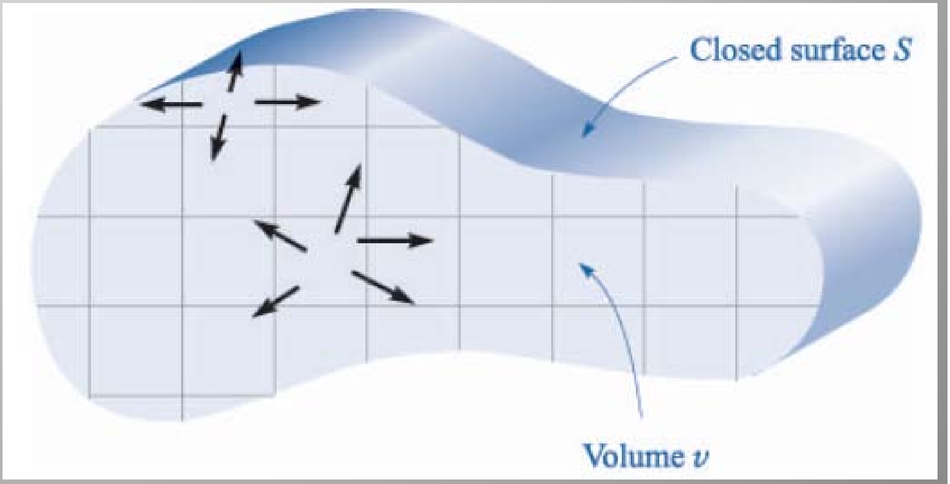
- We shall now give name to a theorem that we actually have obtained, the Divergence Theorem:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D} dv$$

- The first and last terms constitute the divergence theorem:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

"The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface."



The Vector Operator ∇ and The Divergence Theorem

- Example
  Evaluate both sides of the divergence theorem for the field
  D = $2xy\,\mathbf{a}_x + x^2\,\mathbf{a}_y$ C/m^2 and the rectangular parallelepiped fomed by the planes x = 0 and 1, y = 0 and 2, and z = 0 and 3.

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{\text{vol}} \nabla\cdot\mathbf{D}\,dv$$ Divergence Theorem

$$\oint_S \mathbf{D}_S\cdot d\mathbf{S} = \int_0^3\int_0^2 (\mathbf{D})_{x=0}\cdot(-dydz\;\mathbf{a}_x) + \int_0^3\int_0^2 (\mathbf{D})_{x=1}\cdot(dydz\;\mathbf{a}_x)$$
$$+\int_0^3\int_0^1 (\mathbf{D})_{y=0}\cdot(-dxdz\;\mathbf{a}_y) + \int_0^3\int_0^1 (\mathbf{D})_{y=2}\cdot(dxdz\;\mathbf{a}_y)$$

But $(D_x)_{x=0} = 0,\;\; (D_y)_{y=0} = (D_y)_{y=2}$

$$\oint_S \mathbf{D}_S\cdot d\mathbf{S} = \int_0^3\int_0^2 (D_x)_{x=1}\,dydz = \int_0^3\int_0^2 2y\,dydz = \underline{\underline{12\text{ C}}}$$