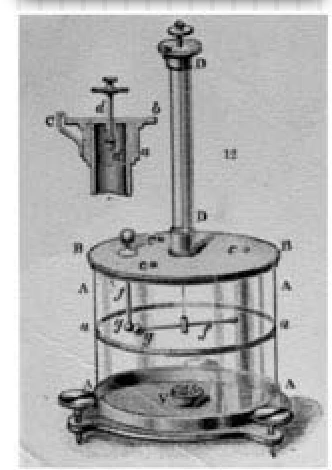
COULOMB'S LAW AND ELECTRIC FIELD INTENSITY

Lecture 2

The Experimental Law of Coulomb

- In 1600, Dr. Gilbert, a physician from England, published the first major classification of electric and non-electric materials.
- He stated that glass, sulfur, amber, and some other materials "not only draw to themselves straw, and chaff, but all metals, wood, leaves, stone, earths, even water and oil."

- In the following century, a French Army Engineer, Colonel Charles Coulomb, performed an elaborate series of experiments using devices invented by himself.
- Coulomb could determine quantitatively the force exerted between two objects, each having a static charge of electricity.
- He wrote seven important treatises on electric and magnetism, developed a theory of attraction and repulsion between bodies of the opposite and the same electrical charge.



The Experimental Law of Coulomb

- Coulomb stated that the force between two very small objects separated in vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them.

$$F = k\frac{Q_1 Q_2}{R^2}$$

- In SI Units, the quantities of charge *Q* are measured in coulombs (C), the separation *R* in meters (m), and the force *F* should be newtons (N).
- This will be achieved if the constant of proportionality *k* is written as:

$$k = \frac{1}{4\pi\varepsilon_0}$$

The Experimental Law of Coulomb

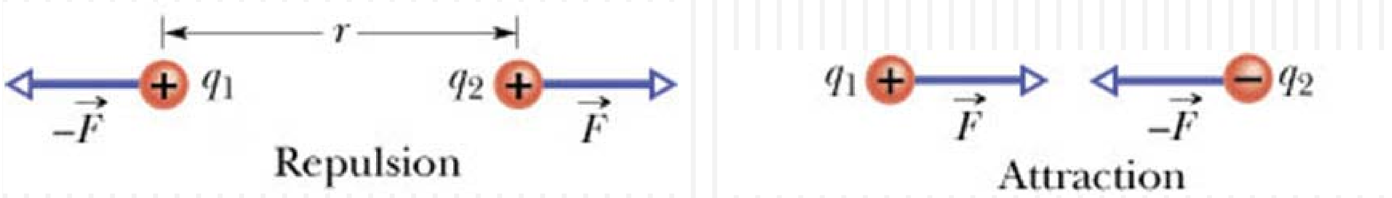
- The *permittivity of free space* ε is measured in farads per meter (F/m), and has the magnitude of:

$$\varepsilon_0 = 8.854 \times 10^{-12} \text{ B } \frac{1}{36\pi} 10^{-9} \text{ F/m}$$

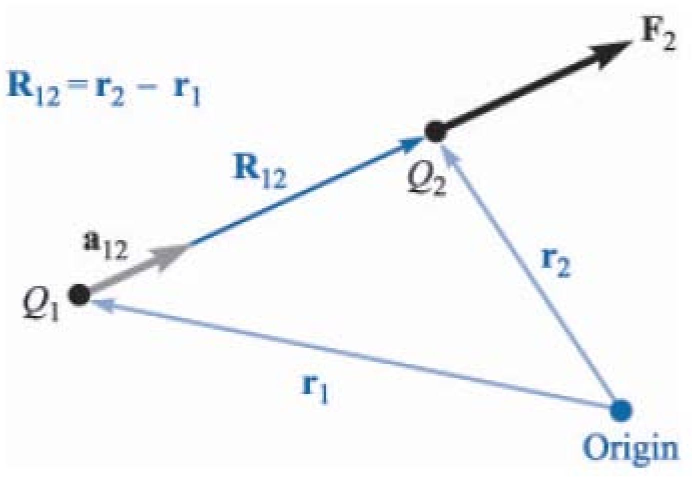
- The Coulomb's law is now:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{R^2}$$

- The force F acts along the line joining the two charges. It is repulsive if the charges are alike in sign and attractive if the are of opposite sign.



The Experimental Law of Coulomb



$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

- In vector form, Coulomb's law is written as:

$$\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R_{12}^2}\mathbf{a}_{12}$$

- F_2 is the force on Q_2, for the case where Q_1 and Q_2 have the same sign, while a_{12} is the unit vector in the direction of R_{12}, the line segment from Q_1 to Q_2.

The Experimental Law of Coulomb

- Example

A charge $Q_1 = 3\times10^{-4}$ C at $M(1,2,3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2,0,5)$ are located in a vacuum. Determine the force exerted on Q_2 by Q_1.

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$$
$$= (2\mathbf{a}_x + 0\mathbf{a}_y + 5\mathbf{a}_z) - (1\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z)$$
$$= 1\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|}$$
$$= \frac{1}{3}(1\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$$

$$\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R_{12}^2}\mathbf{a}_{12}$$
$$= \frac{1}{4\pi(8.854\times10^{-12})}\frac{(3\times10^{-4})(-10^{-4})}{3^2}\frac{1}{3}(1\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$$
$$B\ \underline{\underline{-10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z \text{ N}}}$$

$$\boxed{\mathbf{F}_1 = -\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R_{12}^2}\mathbf{a}_{21}}$$

Electric Field Intensity

- Let us consider one charge, say Q_1, fixed in position in space.
- Now, imagine that we can introduce a second charge, Q_t, as a "unit test charge", that we can move around.
- We know that there exists everywhere a force on this second charge ► This second charge is displaying the existence of a force field.
- The force on it is given by Coulomb's law as:

$$\mathbf{F}_t = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_t}{R_{1t}^2}\mathbf{a}_{1t}$$

- Writing this force as a "force per unit charge" gives:

$$\frac{\mathbf{F}_t}{Q_t} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{R_{1t}^2}\mathbf{a}_{1t}$$

Vector Field,
Electric Field Intensity

Electric Field Intensity

- We define the electric field intensity as the vector of force on a unit positive test charge.
- Electric field intensity, E, is measured by the unit newtons per coulomb (N/C) or volts per meter (V/m).

$$\mathbf{E} = \frac{\mathbf{F}_t}{Q_t} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{R_{1t}^2}\mathbf{a}_{1t}$$

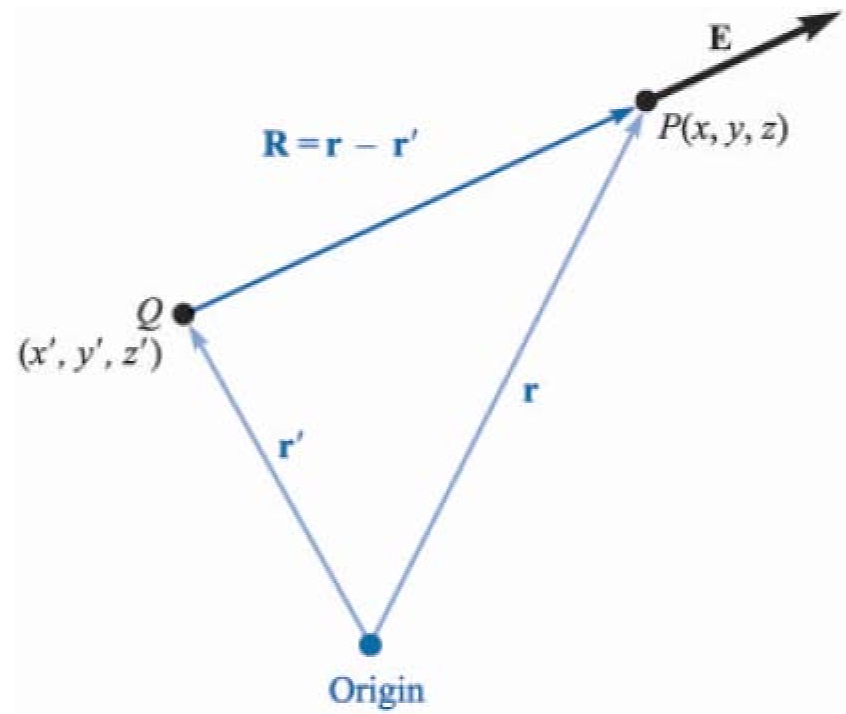
- The field of a single point charge can be written as:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\mathbf{a}_R$$

- a_R is a unit vector in the direction from the point at which the point charge Q is located, to the point at which E is desired/measured.

Electric Field Intensity

- For a charge which is not at the origin of the coordinate, the electric field intensity is:



$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

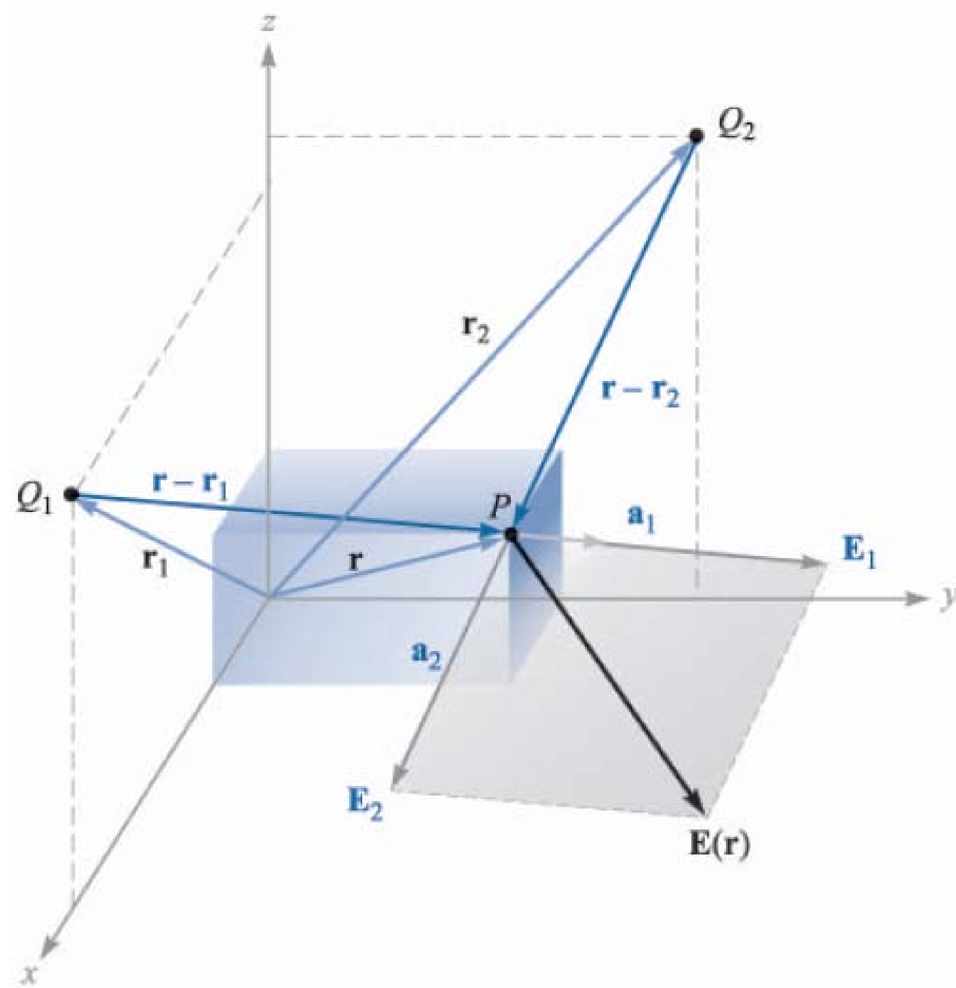
$$= \frac{1}{4\pi\varepsilon_0}\frac{Q(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q\left[(x-x')\mathbf{a}_x+(y-y')\mathbf{a}_y+(z-z')\mathbf{a}_z\right]}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

Electric Field Intensity

- The electric field intensity due to two point charges, say Q_1 at r_1 and Q_2 at r_2, is the sum of the electric field intensity on Q_t caused by Q_1 and Q_2 acting alone (Superposition Principle).

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$



Electric Field Intensity

- Example
  A charge Q_1 of 2 μC is located at at $P_1(0,0,0)$ and a second charge of 3 μC is at $P_2(-1,2,3)$. Find E at $M(3,-4,2)$.

$$\mathbf{r}-\mathbf{r}_1 = 3\mathbf{a}_x - 4\mathbf{a}_y + 2\mathbf{a}_z, \quad |\mathbf{r}-\mathbf{r}_1| = \sqrt{29}$$

$$\mathbf{r}-\mathbf{r}_2 = 4\mathbf{a}_x - 6\mathbf{a}_y - \mathbf{a}_z, \quad |\mathbf{r}-\mathbf{r}_2| = \sqrt{53}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_1(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_2|^3}\right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{2\times10^{-6}(3\mathbf{a}_x - 4\mathbf{a}_y + 2\mathbf{a}_z)}{\left|\sqrt{29}\right|^3} + \frac{3\times10^{-6}(4\mathbf{a}_x - 6\mathbf{a}_y - \mathbf{a}_z)}{\left|\sqrt{53}\right|^3}\right\}$$

$$= \underline{\underline{623.7\mathbf{a}_x - 879.92\mathbf{a}_y + 160.17\mathbf{a}_z \ \ \text{V/m}}}$$

Field Due to a Continuous Volume Charge Distribution

- We denote the volume charge density by ρ_v, having the units of coulombs per cubic meter (C/m^3).
- The small amount of charge ΔQ in a small volume Δv is

$$\Delta Q = \rho_v \Delta v$$

- We may define ρ_v mathematically by using a limit on the above equation:

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

- The total charge within some finite volume is obtained by integrating throughout that volume:

$$Q = \int_{\text{vol}} \rho_v \, dv$$

Field Due to a Continuous Volume Charge Distribution

- Example
  Find the total charge inside the volume indicated by $\rho_v = 4xyz^2$, $0 \le \rho \le 2$, $0 \le \Phi \le \pi/2$, $0 \le z \le 3$. All values are in SI units.

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$

$$\rho_v = 4\cdot\rho\sin\phi\cdot\rho\cos\phi\cdot z^2$$

$$Q = \int_{\text{vol}} \rho_v dv = \int_{z=0}^{3}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{2}(4\cdot\rho\sin\phi\cdot\rho\cos\phi\cdot z^2)(d\rho\cdot\rho d\phi\cdot dz)$$

$$= \int_0^3\int_0^{\pi/2}\int_0^2 4\rho^3 z^2 \sin\phi\cos\phi d\rho d\phi dz$$

$$= \int_0^3\int_0^{\pi/2} 16z^2\sin\phi\cos\phi d\phi dz$$

$$= \int_0^3 8z^2 dz$$

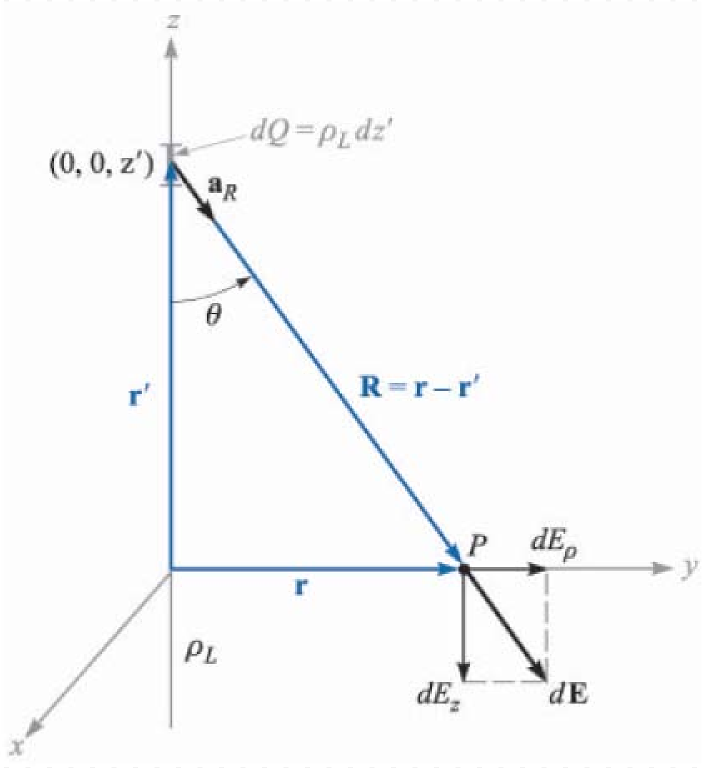
$$= \underline{\underline{72\ \text{C}}}$$

$$\sin 2\phi = 2\sin\phi\cos\phi$$

Field of a Line Charge

- Now we consider a filamentlike distribution of volume charge density. It is convenient to treat the charge as a line charge of density ρ_L C/m.

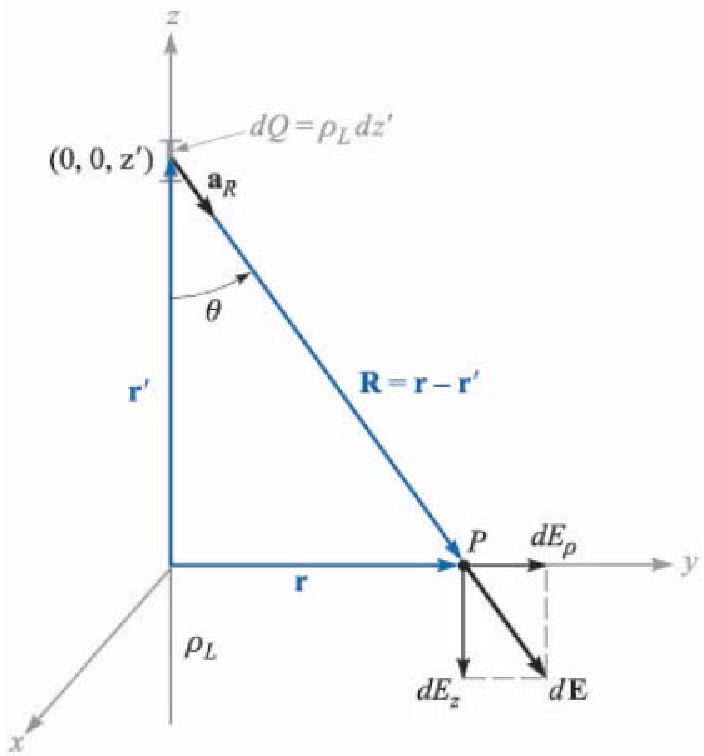
- Let us assume a straight-line charge extending along the *z* axis in a cylindrical coordinate system from
$-\infty$ to $+\infty$.
- We desire the electric field intensity E at any point resulting from a uniform line charge density ρ_L.



$$d\mathbf{E} = dE_\rho \mathbf{a}_\rho + dE_z \mathbf{a}_z$$

Field of a Line Charge

- The incremental field dE only has the components in a_ρ and a_z direction, and no a_Φ direction.
    - Why?
- The component dE_z is the result of symmetrical contributions of line segments above and below the observation point P.
- Since the length is infinity, they are canceling each other ► $dEz = 0$.
- The component dE_ρ exists, and from the Coulomb's law we know that dE_ρ will be inversely proportional to the distance to the line charge, ρ.



$$d\mathbf{E} = dE_\rho \mathbf{a}_\rho + dE_z \mathbf{a}_z$$

Field of a Line Charge

- Take $P(0,y,0)$,

$$\mathbf{r}' = z'\mathbf{a}_z$$

$$\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$$

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

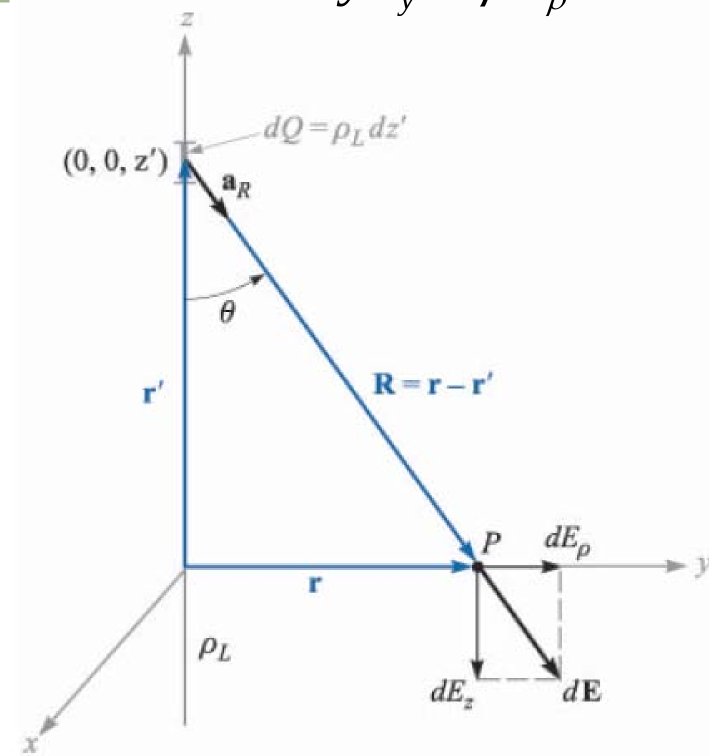
$$= \frac{1}{4\pi\varepsilon_0}\frac{\rho_L dz'(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{(\rho^2+z'^2)^{3/2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\rho_L \rho\mathbf{a}_\rho dz'}{(\rho^2+z'^2)^{3/2}}$$

$$E_\rho = \int_{-\infty}^{+\infty}\frac{1}{4\pi\varepsilon_0}\frac{\rho_L \rho dz'}{(\rho^2+z'^2)^{3/2}}$$

$$= \frac{\rho_L\rho}{4\pi\varepsilon_0}\left[\frac{z'}{\rho^2(\rho^2+z'^2)^{1/2}}\right]_{-\infty}^{+\infty}$$

$$E_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}$$



$$d\mathbf{E} = dE_\rho\mathbf{a}_\rho + dE_z\mathbf{a}_z$$

$$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$$

Field of a Line Charge

- Now let us analyze the answer itself:

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0 \rho}\mathbf{a}_\rho$$

- The field falls off inversely with the distance to the charged line, as compared with the point charge, where the field decreased with the square of the distance.

Field of a Line Charge

- <u>Example D2.5.</u>
  Infinite uniform line charges of 5 nC/m lie along the (positive and negative) *x* and *y* axes in free space. Find E at:
  (*a*) $P_A(0,0,4)$; (*b*) $P_B(0,3,4)$.

$$\mathbf{E}(P_A)=\frac{\rho_{L_x}}{2\pi\varepsilon_0\rho_x}\mathbf{a}_{\rho_x}+\frac{\rho_{Ly}}{2\pi\varepsilon_0\rho_y}\mathbf{a}_{\rho_y}$$

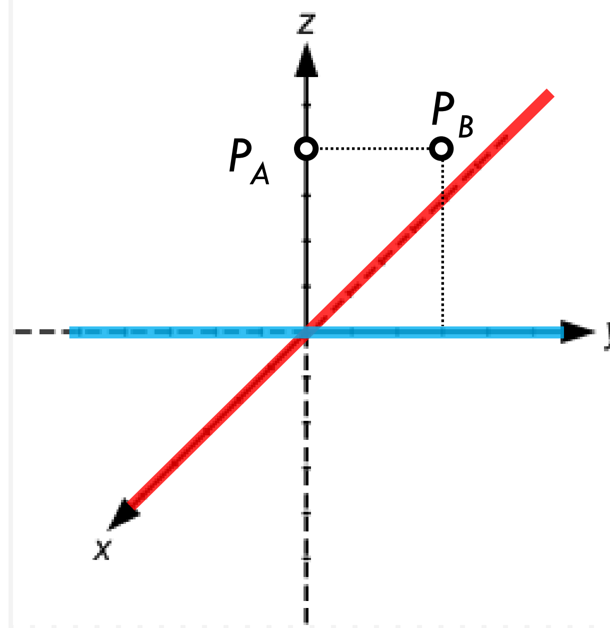
$$=\frac{5\times10^{-9}}{2\pi\varepsilon_0(4)}\mathbf{a}_z+\frac{5\times10^{-9}}{2\pi\varepsilon_0(4)}\mathbf{a}_z$$

$$=\underline{\underline{44.939\mathbf{a}_z\ \text{V/m}}}$$

$$\mathbf{E}(P_B)=\frac{\rho_{L_x}}{2\pi\varepsilon_0\rho_x}\mathbf{a}_{\rho_x}+\frac{\rho_{Ly}}{2\pi\varepsilon_0\rho_y}\mathbf{a}_{\rho_y}$$

$$=\frac{5\times10^{-9}}{2\pi\varepsilon_0(5)}(0.6\mathbf{a}_y+0.8\mathbf{a}_z)+\frac{5\times10^{-9}}{2\pi\varepsilon_0(4}\mathbf{a}_z$$

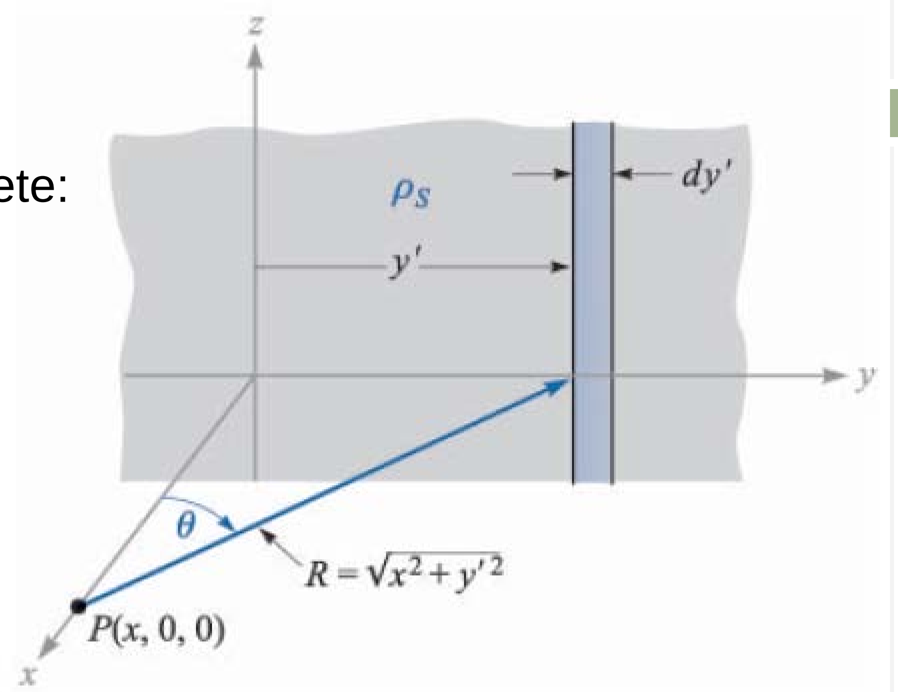
$$=\underline{\underline{10.785\mathbf{a}_y+36.850\mathbf{a}_z\ \text{V/m}}}$$



- ρ is the shortest distance between an observation point and the line charge

Field of a Sheet of Charge

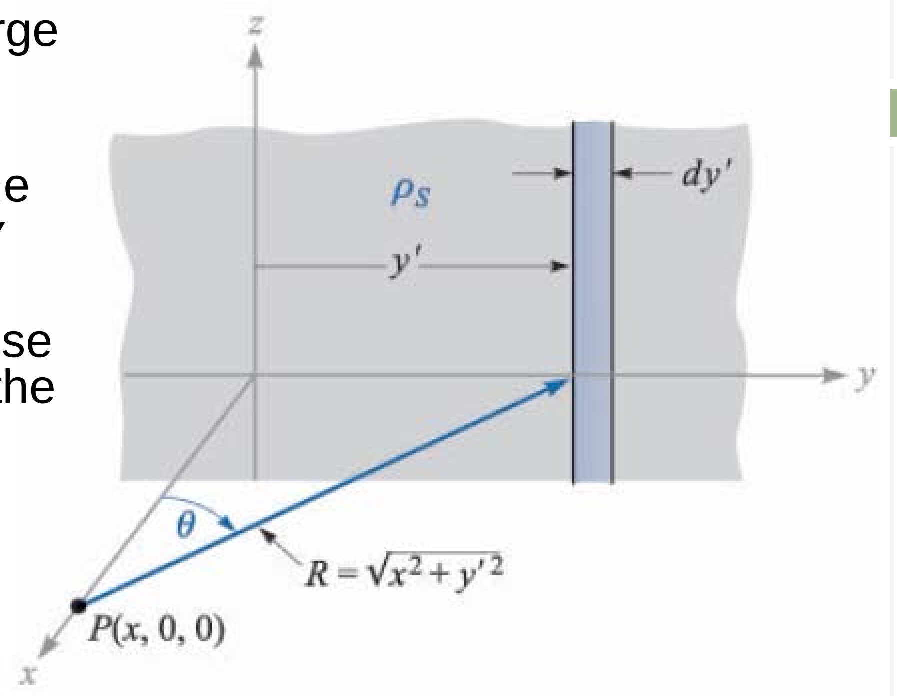
- Another basic charge configuration is the infinite sheet of charge having a uniform density of ρ_S C/m^2.
- The charge-distribution family is now complete: point (Q), line (ρ_L), surface (ρ_S), and volume (ρ_v).



- Let us examine a sheet of charge above, which is placed in the *yz* plane.
- The plane can be seen to be assembled from an infinite number of line charge, extending along the *z* axis, from –∞ to +∞.

Field of a Sheet of Charge

- For a differential width strip *dy*', the line charge density is given by $\rho_L = \rho_S dy'$.
- The component dE_z at *P* is zero, because the differential segments above and below the *y* axis will cancel each other.
- The component dE_y at *P* is also zero, because the differential segments to the right and to the left of *z* axis will cancel each other.



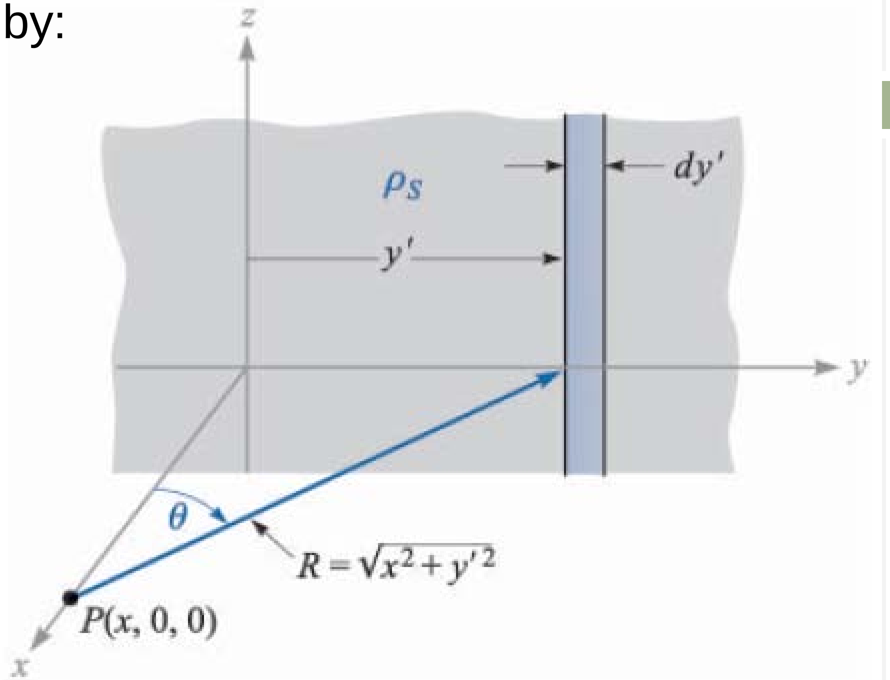
- Only dE_x is present, and this component is a function of *x* alone.

Field of a Sheet of Charge

- The contribution of a strip to E_x at P is given by:

$$dE_x = \frac{\rho_s dy'}{2\pi\varepsilon_0\sqrt{x^2 + y'^2}}\cos\theta$$

$$= \frac{\rho_s}{2\pi\varepsilon_0}\frac{x dy'}{x^2 + y'^2}$$



- Adding the effects of all the strips,

$$E_x = \frac{\rho_s}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{x dy'}{x^2 + y'^2}$$

$$= \frac{\rho_s}{2\pi\varepsilon_0}\tan^{-1}\frac{y'}{x}\Bigg]_{-\infty}^{+\infty}$$

$$= \frac{\rho_s}{2\varepsilon_0}$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

Field of a Sheet of Charge

- Fact: The electric field is always directed away from the positive charge, into the negative charge.
- We now introduce a unit vector a_N, which is normal to the sheet and directed away from it.

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon_0}\mathbf{a}_N$$

- The field of a sheet of charge is constant in magnitude and direction. It is not a function of distance.