Wave Propagation in free space

LECTURE 11

Learning Objectives

- understanding the Wave Propagation in free space
- Understand the wave equations
- Understand the wave velocity
- Explain the *intrinsic impedance*

Introduction

- When considering electromagnetic waves in free space, we note that the medium is *sourceless (ρv =* ***J = 0). Under these conditions, Maxwell's equations may be*** written in terms of **E and H only as**

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1)$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{E} = 0 \quad (3)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (4)$$

Wave Propagation in free space

- We postulate the existence of a *uniform plane wave, in which both fields,* ***E and*** **H, lie in the** ***transverse plane—that is, the plane whose normal is the direction of*** propagation. Furthermore, and by definition, both fields are of constant magnitude in the transverse plane. For this reason, such a wave is sometimes called a *transverse electromagnetic (TEM) wave.*

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t}\mathbf{a}_y \qquad (5)$$

Wave Propagation in free space

- Therefore, in a uniform planewave, the directions of **E and H and the direction of travel are mutually orthogonal. Using the *y-directed magnetic*** field, and the fact that it varies only in *z, simplifies Eq. (1) to read*

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \frac{\partial E_x}{\partial t}\mathbf{a}_x \qquad (6)$$

Wave Propagation in free space

Equations (5) and (6) can be more succinctly written:

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t} \qquad (7)$$

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t} \qquad (8)$$

Wave Propagation in free space

Specifically, we differentiate (7) with respect to z, obtaining:

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \frac{\partial^2 H_y}{\partial t \partial z} \tag{9}$$

Then, (8) is differentiated with respect to t:

$$\frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \tag{10}$$

Substituting (10) into (9) results in

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}} \tag{11}$$

we identify as the waveequation for our *x-polarized TEM electric field in free space.*

Wave Propagation in free space

further identify the propagation velocity:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s} = c \tag{12}$$

where c denotes the velocity of light in free space. A similar procedure, involving differentiating (7) with t and (8) with z, yields the wave equation for the magnetic field; it is identical in form to (11):

$$\frac{\partial^2 H_y}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 H_y}{\partial t^2} \tag{13}$$

As was discussed in Chapter 10, the solution to equations of the form of (11) and (13) will be forward- and backward-propagating waves having the general form [in this case for Eq. (11)]:

Wave Propagation in free space

$$E_x(z,t) = f_1(t - z/\nu) + f_2(t + z/\nu) \quad (14)$$

where again f_1 amd f_2 can be any function whose argument is of the form $t \pm z/\nu$.

From here, we immediately specialize to sinusoidal functions of a specified frequency and write the solution to (11) in the form of forward- and backward-propagating

Wave Propagation in free space

cosines. Because the waves are sinusoidal, we denote their velocity as the *phase velocity*, v_p. The waves are written as:

$$\begin{aligned} E_x(z,t) &= \mathcal{E}_x(z,t) + \mathcal{E}'_x(z,t) \\ &= |E_{x0}| \cos[\omega(t - z/v_p) + \phi_1] + |E'_{x0}| \cos[\omega(t + z/v_p) + \phi_2] \\ &= \underbrace{|E_{x0}| \cos[\omega t - k_0 z + \phi_1]}_{\text{forward } z \text{ travel}} + \underbrace{|E'_{x0}| \cos[\omega t + k_0 z + \phi_2]}_{\text{backward } z \text{ travel}} \end{aligned} \tag{15}$$

In writing the second line of (15), we have used the fact that the waves are traveling in free space, in which case the phase velocity, $v_p = c$. Additionally, the *wavenumber* in free space in defined as

$$k_0 \equiv \frac{\omega}{c} \text{ rad/m} \tag{16}$$

Wave Propagation in free space

We note that *k0 is the phase constant for lossless propagation* of uniform plane waves in free space. The *wavelength in free space is the distance* over which the spatial phase shifts by 2π *radians, assuming fixed time, or*

$$k_0 z = k_0 \lambda = 2\pi \quad \rightarrow \quad \lambda = \frac{2\pi}{k_0} \quad \text{(free space)} \tag{17}$$

$$\omega t - k_0 z = \omega(t - z/c) = 2m\pi \tag{18}$$

Wave Propagation in free space

we express the real instantaneous fields of Eq. (15) in terms of their phasor forms. Using the forward-propagating field in (15), we write:

$$\mathcal{E}_x(z,t) = \frac{1}{2}\underbrace{|E_{x0}|e^{j\phi_1}}_{E_{x0}} e^{-jk_0 z} e^{j\omega t} + c.c. = \frac{1}{2}E_{xs}e^{j\omega t} + c.c. = \text{Re}[E_{xs}e^{j\omega t}] \tag{19}$$

Wave Propagation in free space

Example:

Given the complex amplitude of the electric field of a uniform plane wave, $\mathbf{E}_0 = 100\mathbf{a}_x + 20\angle 30^\circ \mathbf{a}_y$ V/m, construct the phasor and real instantaneous fields if the wave is known to propagate in the forward z direction in free space and has frequency of 10 MHz.

Solution. We begin by constructing the general phasor expression:

$$\mathbf{E}_s(z) = \left[100\mathbf{a}_x + 20e^{j30^\circ}\mathbf{a}_y\right] e^{-jk_0 z}$$

where $k_0 = \omega/c = 2\pi \times 10^7/3 \times 10^8 = 0.21$ rad/m. The real instantaneous form is then found through the rule expressed in Eq. (19):

$$\begin{aligned}\mathcal{E}(z,t) &= \text{Re}\left[100e^{-j0.21z}e^{j2\pi\times10^7 t}\mathbf{a}_x + 20e^{j30^\circ}e^{-j0.21z}e^{j2\pi\times10^7 t}\mathbf{a}_y\right] \\ &= \text{Re}\left[100e^{j(2\pi\times10^7 t-0.21z)}\mathbf{a}_x + 20e^{j(2\pi\times10^7 t-0.21z+30^\circ)}\mathbf{a}_y\right] \\ &= 100\cos(2\pi\times10^7 t - 0.21z)\mathbf{a}_x + 20\cos(2\pi\times10^7 t - 0.21z + 30^\circ)\,\mathbf{a}_y\end{aligned}$$

Wave Propagation in free space

It is evident that taking the partial derivative of any field quantity with respect to time is equivalent to multiplying the corresponding phasor by $j\omega$. As an example, we can express Eq. (8) (using sinusoidal fields) as

$$\frac{\partial \mathcal{H}_y}{\partial z} = -\epsilon_0 \frac{\partial \mathcal{E}_x}{\partial t} \qquad (20)$$

where, in a manner consistent with (19):

$$\mathcal{E}_x(z,t) = \frac{1}{2} E_{xs}(z)\, e^{j\omega t} + c.c. \qquad \text{and} \qquad \mathcal{H}_y(z,t) = \frac{1}{2} H_{ys}(z)\, e^{j\omega t} + c.c. \qquad (21)$$

On substituting the fields in (21) into (20), the latter equation simplifies to

$$\boxed{\frac{dH_{ys}(z)}{dz} = -j\omega\epsilon_0 E_{xs}(z)} \qquad (22)$$

Wave Propagation in free space

We next apply this result to Maxwell's equations, to obtain them in phasor form. Substituting the field as expressed in (21) into Eqs. (1) through (4) results in

$$\nabla \times \mathbf{H}_s = j\omega\epsilon_0 \mathbf{E}_s \tag{23}$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu_0 \mathbf{H}_s \tag{24}$$

$$\nabla \cdot \mathbf{E}_s = 0 \tag{25}$$

$$\nabla \cdot \mathbf{H}_s = 0 \tag{26}$$

It should be noted that (25) and (26) are no longer independent relationships, for they can be obtained by taking the divergence of (23) and (24), respectively.

Eqs. (23) through (26) may be used to obtain the sinusoidal steady-state vector form of the wave equation in free space. We begin by taking the curl of both sides of (24):

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu_0 \nabla \times \mathbf{H}_s = \nabla(\nabla \cdot \mathbf{E}_s) - \nabla^2 \mathbf{E}_s \tag{27}$$

where the last equality is an identity, which defines the *vector Laplacian* of $\mathbf{E}_s$:

$$\nabla^2 \mathbf{E}_s = \nabla(\nabla \cdot \mathbf{E}_s) - \nabla \times \nabla \times \mathbf{E}_s$$

Wave Propagation in free space

From (25), we note that $\nabla \cdot \mathbf{E}_s = 0$. Using this, and substituting (23) in (27), we obtain

$$\nabla^2 \mathbf{E}_s = -k_0^2 \mathbf{E}_s \tag{28}$$

Equation (28) is known as the vector Helmholtz equation in free space

Wave Propagation in free space

The *x component of (28) becomes, still using the* del-operator notation

$$\nabla^2 E_{xs} = -k_0^2 E_{xs} \tag{29}$$

and the expansion of the operator leads to the second-order partial differential equation

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} = -k_0^2 E_{xs}$$

Again, assuming a uniform plane wave in which E_{xs} does not vary with x or y, the two corresponding derivatives are zero, and we obtain

$$\frac{d^2 E_{xs}}{dz^2} = -k_0^2 E_{xs} \tag{30}$$

the solution of which we already know:

$$E_{xs}(z) = E_{x0} e^{-jk_0 z} + E'_{x0} e^{jk_0 z} \tag{31}$$

Wave Propagation in free space

Let us now return to Maxwell's equations, (23) through (26), and determine the form of the **H** field. Given $\mathbf{E}_s$, $\mathbf{H}_s$ is most easily obtained from (24):

$$\nabla \times \mathbf{E}_s = -j\omega\mu_0 \mathbf{H}_s \tag{24}$$

which is greatly simplified for a single E_{xs} component varying only with z,

$$\frac{dE_{xs}}{dz} = -j\omega\mu_0 H_{ys}$$

Using (31) for E_{xs}, we have

$$H_{ys} = -\frac{1}{j\omega\mu_0}\left[(-jk_0)E_{x0}e^{-jk_0z} + (jk_0)E'_{x0}e^{jk_0z}\right]$$

$$= E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{-jk_0z} - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{jk_0z} = H_{y0}e^{-jk_0z} + H'_{y0}e^{jk_0z} \tag{32}$$

In real instantaneous form, this becomes

$$H_y(z,t) = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t - k_0 z) - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t + k_0 z) \tag{33}$$

where E_{x0} and E'_{x0} are assumed real.

Wave Propagation in free space

In general, we find from (32) that the electric and magnetic field amplitudes of the forward-propagating wave in free space are related through

$$E_{x0} = \sqrt{\frac{\mu_0}{\epsilon_0}} H_{y0} = \eta_0 H_{y0} \tag{34a}$$

We also find the backward-propagating wave amplitudes are related through

$$E'_{x0} = -\sqrt{\frac{\mu_0}{\epsilon_0}} H'_{y0} = -\eta_0 H'_{y0} \tag{34b}$$

where the *intrinsic impedance* of free space is defined as

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \doteq 120\pi \ \Omega \tag{35}$$