Lecture 3

The Multiplication Rules and Conditional Probability:

The Multiplication Rules

The *multiplication rules* can be used to find the probability of two or more events that occur in sequence. For example, if you toss a coin and then roll a die, you can find the probability of getting a head on the coin *and* a 4 on the die.

These two events are said to be *independent* since the outcome of the first event (tossing a coin) does not affect the probability outcome of the second event (rolling a die).

Two events *A* and *B* are independent events if the fact that *A* occurs does not affect the probability of *B* occurring.

To find the probability of two independent events that occur in sequence, you must

find the probability of each event occurring separately and then multiply the answers.

For example, if a coin is tossed twice, the probability of getting two heads is $1/2 \times 1/2 = 1/4$.

This result can be verified by looking at the sample space HH, HT, TH, TT. Then $P(HH) = 1/4$

Multiplication Rule 1

When two events are independent, the probability of both occurring is:

$$P(A \text{ and } B) = P(A) \cdot P(B$$

Example 1:

Tossing a Coin:

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution:

P(head and 4) = P(head) . P(4)= 1/2x1/6=1/12

Note that the sample space for the coin is H, T; and for the die it is 1, 2, 3, 4, 5, 6.

Example2:

Selecting a Colored Ball

A box contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its

color noted. Then it is replaced. A second ball is selected and its color noted. Find the probability of each of these.

a. Selecting 2 blue balls

b. Selecting 1 blue ball and then 1 white ball

c. Selecting 1 red ball and then 1 blue ball

Solution:

a. *P*(blue and blue) = *P*(blue) . *P*(blue)= 2/10 x2/10=4/100= 1/25

b. *P*(blue and white)= *P*(blue) . *P*(white)= 2/10 x5/10 = 10/100 =1/10

c. *P*(red and blue) = *P*(red) . *P*(blue) = 3/10 x2/10 =6/100 = 3/50

Multiplication rule 1 can be extended to three or more independent events by using the formula

$$P(A \text{ and } B \text{ and } C \text{ and } \ldots \text{ and } K) = P(A) \cdot P(B) \cdot P(C) \ldots P(K)$$

When a small sample is selected from a large population and the subjects are not replaced, the probability of the event occurring changes so slightly that for the most part, it is considered to remain the same.

Example3 :Survey on Stress

A Harris poll found that 46% of Americans say they suffer great stress at least once a

week. If three people are selected at random, find the probability that all three will say that they suffer great stress at least once a week.

Solution

Let S denote stress. Then

$$P(S \text{ and } S \text{ and } S) = P(S) \cdot P(S) \cdot P(S)$$

$$(0.46)(0.46)(0.46) = 0.097$$

Example4 : Male Color Blindness

Approximately 9% of men have a type of color blindness that prevents them from distinguishing between red and green. If 3 men are selected at random, find the probability that all of them will have this type of red-green color blindness.

Solution

Let C denote red-green color blindness. Then

$$P(\text{C and C and C}) = P(\text{C}) \cdot P(\text{C}) \cdot P(\text{C})$$

$$(0.09)\text{x}(0.09)\text{x}(0.09) = 0.000729$$

Hence, the rounded probability is 0.0007.

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In Examples 1 through 4, the events were independent of one another, since

the occurrence of the first event in no way affected the outcome of the second event.

On the other hand, when the occurrence of the first event changes the probability of the occurrence of the second event, the two events are said to be *dependent*.

For example, suppose a card is drawn from a deck and *not* replaced, and then a second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Before an answer to the question can be given, you must realize that the events are dependent.

The probability of selecting an ace on the first draw is 4/52 . If that card is *not* replaced, the probability of selecting a king on the second card is 4/51 , since there are 4 kings

and 51 cards remaining. The outcome of the first draw has affected the outcome of the second draw.

Dependent events are formally defined now.

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be dependent events.

Here are some examples of dependent events:

- Drawing a card from a deck, not replacing it, and then drawing a second card.
- Selecting a ball from a box not replacing it, and then selecting a second ball.
- Having high grades and getting a scholarship.
- Parking in a no-parking zone and getting a parking ticket.

To find probabilities when events are dependent, use the multiplication rule with a

modification in notation. For the problem just discussed, the probability of getting an ace

on the first draw is 4/52 , and the probability of getting a king on the second draw is 4/51 . By the multiplication rule, the probability of both events occurring is=

$$4/52 \times 4/51 = 16/2652 = 4/663$$

The event of getting a king on the second draw *given* that an ace was drawn the first time is called a *conditional probability.*

The conditional probability of an event *B* in relationship to an event *A* is the probability

that event *B* occurs after event *A* has already occurred. The notation for conditional

probability is $P(B/A)$. This notation does not mean that *B* is divided by *A*; rather, it means the probability that event *B* occurs given that event *A* has already occurred.

In the card example, $P(B/A)$ is the probability that the second card is a king given that the first card is an ace, and it is equal to 4/51 since the first card was *not* replaced.

Multiplication Rule 2

When two events are dependent, the probability of both occurring is=

$$P(A \text{ and } B) = P(A) \cdot P(B/A)$$

Example :University Crime

At a university in western Pennsylvania, there were 5 burglaries reported in 2003, 16 in 2004, and 32 in 2005. If a researcher wishes to select at random two burglaries to further investigate, find the probability that both will have occurred in 2004.

Solution

In this case, the events are dependent since the researcher wishes to investigate two distinct cases. Hence the first case is selected and not replaced.

$P(C_1 \text{ and } C_2) = P(C_1) \,.\, P(C_2/C_1) = 16/53 \times 15/52 = 60/689$

Example : Drawing Cards

Three cards are drawn from an ordinary deck and not replaced. Find the probability of

these events.

a. **Getting 3 jacks**

b. **Getting an ace, a king, and a queen in order**

c. Getting a club, a spade, and a heart in order

d. **Getting 3 clubs**

Solution

a. P(3 jacks) = 4/52. 3/51. 2/50 = 24/132600
b. P(ace and king and queen)= 4/52 . 4/51. 4/50 = 64/132600 = 8/16575
c. P(club and spade and heart)= 13/52 . 13/51.13/50 = 2197/132600 =169/10200
d. P(3 clubs)= 13/52 .12/51. 11/50 = 1716/132600 = 11/850

Conditional Probability

The conditional probability of an event *B* in relationship to an event *A* was defined as the

probability that event *B* occurs after event *A* has already occurred.

The conditional probability of an event can be found by dividing both sides of the equation for multiplication rule 2 by *P*(*A*), as shown:

$$P(A \text{ and } B) = P(A) \cdot P(B/A)$$

$$\frac{P(A \text{ and } B)}{P(A)} = \frac{P(A) \cdot P(B/A)}{p(A)} =$$

$$\frac{P(A \text{ and } B)}{P(A)} = P(B/A)$$

Formula for Conditional Probability

The probability that the second event *B* occurs given that the first event *A* has occurred can be

found by dividing the probability that both events occurred by the probability that the first

event has occurred. The formula is

$$P(B/A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example :Selecting Colored Chips

A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip *and* a white chip is=15/56 , and the probability of selecting a black chip on the first draw is=3/8 , find the probability of selecting the white chip on the second draw, *given* that the first chip selected was a black chip.

Solution:

Let :*B* selecting a black chip *W* selecting a white chip

Then: $P(W/B) = P(B \text{ and } W) / P(B) = \frac{15/56}{3/8} = 5/7$

Example Parking Tickets

The probability that Sam parks in a no-parking zone *and* gets a parking ticket is 0.06,

and the probability that Sam cannot find a legal parking space and has to park in the no parking zone is 0.20. On Tuesday, Sam arrives at school and has to park in a no-parking

zone. Find the probability that he will get a parking ticket.

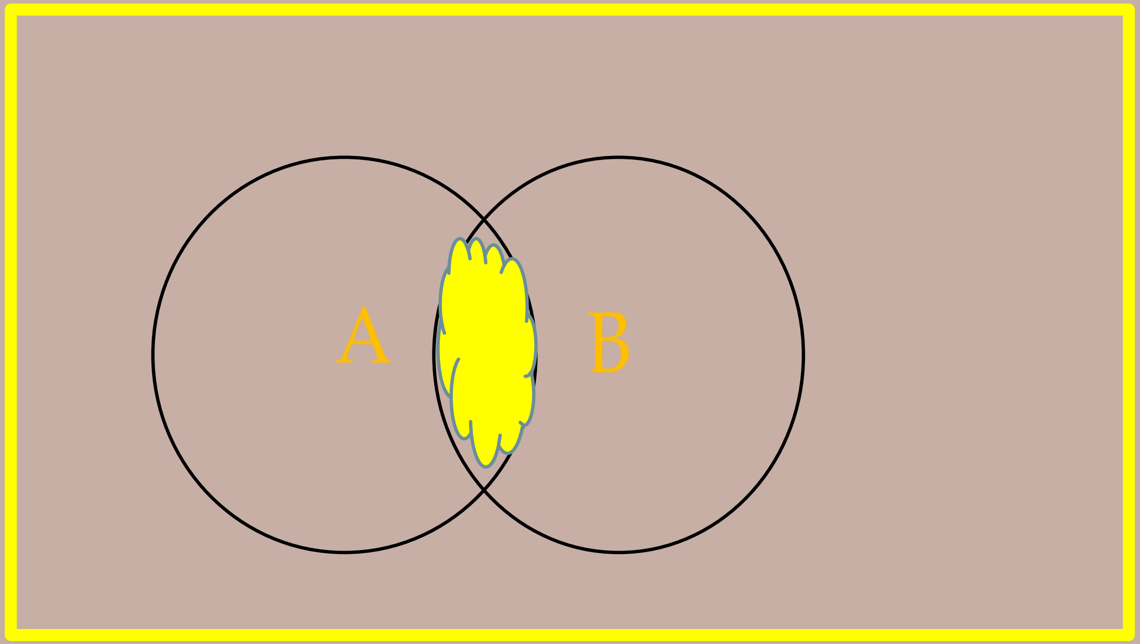
Solution:

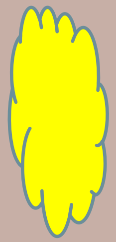
Let :*N* = parking in a no-parking zone *T*= getting a ticket

$$P(T/N) = P(N \text{ and } T) / P(N) = 0.06 / 0.20 = 0.30$$

Hence, Sam has a 0.30 probability of getting a parking ticket, given that he parked in a no-parking zone.

Venn Diagram for Conditional Probability



P(A and B) =A n B 

$$P(B/A) = \frac{P(A \text{ and } B)}{P(A)}$$

Probabilities for “At Least”

The multiplication rules can be used with the complementary event rule

$P(\bar{A}) = 1 - P(A)$ or $P(A) = 1 - P(\bar{A}) = P(A) + P(\bar{A}) = 1$

to simplify solving probability problems involving “at least.”

Example Drawing Cards

A game is played by drawing 4 cards from an ordinary deck and replacing each card after it is drawn. Find the probability that at least 1 ace is drawn.

Solution

It is much easier to find the probability that no aces are drawn (i.e., losing) and then subtract that value from 1 than to find the solution directly, because that would involve finding the probability of getting 1 ace, 2 aces, 3 aces, and 4 aces and then adding the results.

Let A at least 1 ace is drawn and $\bar{A}$ no aces drawn. Then

$P(\overline{A})$= 48/52 .48/52 .48/52 .48/52 =

12/13 .12/13 .12/13 .12/13 =20736/28561

Hence,

$$P(A) = 1 - P(\overline{A})$$

$$P(\text{winning}) = 1 - P(\text{losing}) = 1 - \frac{20{,}736}{28{,}561} = \frac{7825}{28{,}561} = 0.27$$

or a hand with at least 1 ace will occur about 27% of the time.

Example Tossing Coins

A coin is tossed 5 times. Find the probability of getting at least 1 tail.

Solution

It is easier to find the probability of the complement of the event, which is "all heads,"

and then subtract the probability from 1 to get the probability of at least 1 tail.

$$P(A) = 1 - P(\bar{A})$$

$$P(\text{at least 1 tail}) = 1 - P(\text{all heads})$$

$$P(\text{all heads}) = (\tfrac{1}{2})^5 = 1/32$$

Hence,

$$P(\text{at least 1 tail}) = 1 - 1/32 = 31/32$$

Exercises

a. Tossing a coin and drawing a card from a deck

Independent

b. Drawing a ball from an urn, not replacing it, and then drawing a second ball

Dependent

c. Getting a raise in salary and purchasing a new car

Dependent

d. Driving on ice and having an accident

Dependent

e. Having a large shoe size and having a high IQ
Independent

f. A father being left-handed and a daughter being left-handed

Dependent

g. Smoking excessively and having lung cancer
Dependent

h. Eating an excessive amount of ice cream and smoking an excessive amount of cigarettes

Independent