NUSU

Faculty of Administrative Sciences

Business Mathematics-2 Semester: 2

LC(6)

THE PRODUCT AND QUOTIENT RULES

Theorem (The Product Rule)

If f and g are differentiable at x, then so is the product $f \cdot g$,and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Example 1

Find $\frac{dy}{dx}$ if $y = (4x^2 - 1)(7x^3 + x)$.

Solution

There are two methods that can be used to find $\frac{dy}{dx}$. We can either use the product rule or we can multiply out the factors in y and then differentiate. We will give both methods.

Method 1. (Using the Product Rule)

$$\frac{dy}{dx} = \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)]$$

$$= (4x^2 - 1)\frac{d}{dx}[7x^3 + x] + (7x^3 + x)\frac{d}{dx}[4x^2 - 1]$$

$$= (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x) = 140x^4 - 9x^2 - 1$$

Method 2. (Multiplying First)

$$y = (4x^2 - 1)(7x^3 + x) = 28x^5 - 3x^3 - x$$

Thus

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

Which agrees with the result obtained using the product rule.

Example 2

Find $\frac{ds}{dt}$ if $s = (1 + t)\sqrt{t}$.

Solution

Applying the product rule yields

$$\frac{ds}{dt} = \frac{d}{dt}\left[(1 + t)\sqrt{t}\right] = (1 + t)\frac{d}{dt}\left[\sqrt{t}\right] + \sqrt{t}\frac{d}{dt}[1 + t]$$

$$= \frac{1 + t}{2\sqrt{t}} + \sqrt{t} = \frac{1 + 3t}{2\sqrt{t}}$$

Theorem (The Quotient Rule)

If f and g are both differentiable at x and if $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example 1

Find $y'(x)$ for $y = \frac{x^3+2x^2-1}{x+5}$

Solution. Applying the quotient rule yields

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3+2x^2-1}{x+5}\right]$$

$$= \frac{(x+5)\frac{d}{dx}[x^3+2x^2-1]-(x^3+2x^2-1)\frac{d}{dx}[x+5]}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x)-(x^3+2x^2-1)(1)}{(x+5)^2}$$

$$= \frac{(3x^3+19x^2+20x)-(x^3+2x^2-1)}{(x+5)^2}$$

$$= \frac{2x^3+17x^2+20x+1}{(x+5)^2}$$

Example 2

Find $f'(x)$ for $f(x) = \frac{x^2-1}{x^4+1}$

Solution

$$f'(x) = \frac{d}{dx}\left[\frac{x^2-1}{x^4+1}\right]$$

$$= \frac{(x^4+1)\frac{d}{dx}[x^2-1]-(x^2-1)\frac{d}{dx}[x4+1]}{(x^4+1)^2}$$

$$= \frac{(x^4+1)2x-(x^2-1)4x^3}{(x^4\ +1)^2}$$

.

$$= \frac{-2x^5+4x^3+2x}{(x^4\ +1)^2} = -\frac{2x(x^4-2x^2-1)}{(x^4\ +1)^2}$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The main objective of this section is to obtain formulas for the derivatives of the six basic trigonometric functions. If needed, you will find a review of trigonometric functions in Appendix B.

We will assume in this section that the variable x in the trigonometric functions sinx, cosx, tanx, cotx, secx,and cscx is measured in radians. Also,we will need the limits but restated as follows using h rather than x as the variable:

$$\lim_{h\to 0} \frac{sinh}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{1-\cos h}{h} = 0$$

Let us start with the problem of differentiating $f(x) = sinx$. Using the definition of the derivative we obtain

$$f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-sinx}{h}$$

$$= \lim_{h\to 0} \frac{sinx\ cosh + cosx\ sinh - sinx}{h}$$

By the addition formula for sine

$$= \lim_{h \to 0} \frac{sinx(cosh - 1) + cosxsinh}{h}$$

$$= \lim_{h \to 0} \frac{cosxsinh - sinx(1 - cosh)}{h}$$

Algebraic reorganization

$$= \lim_{h \to 0} cosx \,.\, \lim_{h \to 0} \frac{sinh}{h} - \lim_{h \to 0} sinx \,.\, \lim_{h \to 0} \frac{(1 - cosh)}{h}$$

$$= \lim_{h \to 0} cosx \,.\,(1) - \lim_{h \to 0} sinx \,.\,(0) = \lim_{h \to 0} cosx = cosx$$

Thus, we have shown that

$$\frac{d}{dx}[sinx] = cosx \qquad (1)$$

In the exercises we will ask you to use the same method to derive the following formula for the derivative of $cosx$:

$$\frac{d}{dx}[cosx] = -sinx \qquad (2)$$

The derivatives of the remaining trigonometric functions are

$$\frac{d}{dx}[tanx] = \sec^2 x \qquad \frac{d}{dx}[secx] = secx.tanx$$

$$\frac{d}{dx}[cotx] = -\csc^2 x \qquad \frac{d}{dx}[cscx] = -cscx.cotx$$

These can all be obtained using the definition of the derivative, but it is easier to use Formulas (1) and (2) and apply the quotient rule to the relationships

$$tanx = \frac{sinx}{cosx} \ , \quad cotx = \frac{cosx}{sinx} \ , \quad secx = \frac{1}{cosx} \ , \ cscx = \frac{1}{sinx}$$

For example,

$$\frac{d}{dx}[tanx] = \frac{d}{dx}\left(\frac{sinx}{cosx}\right)$$

$$= \frac{cosx\frac{d}{dx}(sinx) - sinx\frac{d}{dx}(cosx)}{cos^2x}$$

$$= \frac{cosx.cosx - sinx(-sinx)}{cos^2x}$$

$$= \frac{\cos^2 x - \sin^2 x}{cos^2x} = \frac{1}{cos^2x} = cec^2x$$

General rules

$\frac{d}{dx}(sin(ax))$	$acos(ax)$
$\frac{d}{dx}(cos(ax))$	$-asin(ax)$
$\frac{d}{dx}(tan(ax))$	$a\sec^2(ax)$

$\frac{d}{dx}(cot(ax))$	$-acosec^2(ax)$
$\frac{d}{dx}(sec(ax))$	$asec(ax).tan(ax)$
$\frac{d}{dx}(csc(ax))$	$-acsc(ax).cot(ax)$

DERIVATIVES OF EXPONENTIAL FUNCTIONS

Our next objective is to show that the general exponential function b^x $(b > 0, b \neq 1)$ is differentiable everywhere and to find its derivative. To do this, we will use the fact that b^x is the inverse of the function $f(x) = \log_b x$. We will assume that $b > 1$. With this assumption we have $\ln b > 0$, so

$$f'(x) = \frac{d}{dx}[\log_b x] = \frac{1}{xlnb} > 0$$

for all x in the interval $(0, +\infty)$ It now follows fromTheorem that $f^{-1}(x) = b^x$ is differentiable for all x in the range of $f(x) = \log_b x$. But we know from Table that the range of $\log_b x$ is $(-\infty, +\infty)$, so we have established that $\hat{}b\ x$ is differentiable everywhere.

To obtain a derivative formula for b^x we rewrite

$y\ = b^x\ \ as\ x\ = \log_b y$ and differentiate implicitly obtain

$1 = \frac{1}{ylnb} \cdot \frac{dy}{dx}$ Solving for $\frac{dy}{dx}$ and replacing y by b^x we have

$\frac{dy}{dx} = y\ lnb = b^x lnb$

Thus, we have shown that

$$\frac{d}{dx}[b^x] = b^x lnb \qquad (1)$$

In the special case where $b = e$ we have $lne = 1$, so that (1) becomes

$$\frac{d}{dx}[e^x] = e^x \qquad (2)$$

Moreover, if u is a differentiable function of x, then it follows from (1) and (2) that

$$\frac{d}{dx}[bu] = b^u\ lnb \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

Example

$Find\ \frac{dy}{dx}\ if$

1) $y = \cos(5x)$, 2) $y = sec4x$, 3) $y = x.sinx$,

4) $y = \frac{x}{x^2+1}$, 5) $y = e^{2x}$, 6) $y = \frac{e^{3x}}{cosx}$

Solution

1) $\frac{dy}{dx} = \frac{d}{dx}(\cos(5x)) = -5\sin(5x)$

2) $\frac{dy}{dx} = \frac{d}{dx}(sec(4x)) = 4\sec(4x)\tan(4x)$

3) $\frac{dy}{dx} = \frac{d}{dx}(x.sinx) = x.\frac{d}{dx}(sinx) + sinx.\frac{d}{dx}(x)$

$= x.cosx + sinx$

4) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x^2+1}\right)$

$$= \frac{(x^2+1)\frac{d}{dx}(x) - x.\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

5) $\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = 2e^{2x}$

6) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^{3x}}{cosx}\right) = \frac{cosx\frac{d}{dx}(e^{3x}) - e^{3x}\frac{d}{dx}(cosx)}{(cosx)^2}$

$$= \frac{cosx.3e^{3x} - e^{3x}(-sinx)}{(cosx)^2}$$

$$= \frac{e^{3x}(3cosx + sinx)}{(cosx)^2}$$

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$Find\ \frac{dy}{dx}$ for the following functions

1) $y = \frac{x^3+4x}{2x-2}$
2) $y = tan(2x)$
3) $y = (x+1).\csc(3x)$
4) $y = x.cosx$
5) $y = \frac{x^2-2x}{x+1}$
6) $y = e^{3x}$
7) $y = x.e^x$,
8) $y = \frac{e^{3x}}{x}$